2017

A Homotopy-Theoretic Approach to the Topological Tverberg Conjecture

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Submitted in Partial Fulfillment
of the Prerequisite for Honors
in the Department of Mathematics

April 2017

Abstract

Let $r \geq 2$ and $d \geq 1$ be integers, let $N = (d+1)(r-1)$, and let Δ^N denote a standard N-simplex. The Topological Tverberg Conjecture states that any continuous map $f : \Delta^N \to \mathbb{R}^d$ has r-fold self-intersections such that the preimages of the r-fold intersection points come from pairwise disjoint faces in the original simplex. F. Frick recently announced a counterexample to the conjecture for $d \geq 3r+1$, when r is not a power of a prime. This thesis will discuss an alternative analysis of Frick's counterexample using the manifold calculus of functors. We hope that this technique will provide insight into other counterexamples to the Topological Tverberg Conjecture.

Acknowledgements

I would first like to thank my advisor, Professor Ismar Volić for his guidance throughout this project. Thank you for encouraging me when I thought that this project was impossible, and for answering my frantic emails. I would also like to thank the other members of my committee: Professor Karen Lange, Professor Philip Hirschhorn, and Professor Mary Kate McGowan. Not only have you volunteered your time to read this thesis, but each of you has also provided invaluable support and advice during my time at Wellesley.

Thank you also to my friends, who have stood by me throughout my thesis process and my Wellesley experience. I am so grateful for all of the hugs, pep-talks, and late-night phone calls. I would also like to thank my family, for their unwavering love and encouragement.

Lastly, I would like to thank the Brachman Hoffman Fellowship for supporting the summer research that was the foundation for this thesis.

Contents

Introduction

The Topological Tverberg Conjecture has been an unsolved question in combinatorial topology since the 1980s. The conjecture was originally stated by Bárány, Shlosman, and Szűcs in [**2**]. Let $r \geq 2$ and $d \geq 1$ be integers, and $\Delta^{(d+1)(r-1)}$ be a standard $(d+1)(r-1)$ simplex. Then the Topological Tverberg Conjecture states that any continuous map $f : \Delta^{(d+1)(r-1)} \to \mathbb{R}^d$ has r-fold self-intersections such that the preimages of the r-fold intersection points come from pairwise disjoint faces in the original simplex. The conjecture is true in cases where r is a power of a prime ([**7**]), but is false when r is not a power of a prime and $d \geq 2r+1$. However, the conjecture is still open for values of $d < 12$.

Previous counterexamples for r not a power of a prime have been found using, among other techniques, a higher-dimensional analog of the Whitney Trick (e.g. [**4**], [**1**]). The original Whitney Trick removes two-fold intersections from maps under certain conditions. The higher-dimensional version removes r-fold intersections from maps, thus producing counterexamples to the Topological Tverberg Conjecture. Since the original Whitney Trick can be recovered using the manifold calculus of functors, our work aims to recover the higher-dimensional Whitney Trick using the same technique. Our goal is to recover counterexamples to the Topological Tverberg Conjecture in an algebraic topology setting.

The manifold calculus of functors approximates functors (maps between categories) via a Taylor tower of approximations, which is analogous to a Taylor polynomial in single-variable calculus. The space of embeddings of a manifold M in another manifold N can be viewed as a (contravariant) functor from the category of open subsets of M to the category of topological spaces. Applying the manifold calculus to this functor is a way to recover the Whitney Trick, which indicates that the manifold calculus of functors could be useful in studying the Topological Tverberg Conjecture. More precisely, in [**4**], F. Frick defines an *almost r-immersion*, which is a map f from a simplicial complex K to $\mathbb{R}^d$ such that the images of any r pairwise disjoint faces of K share no common points. The Topological Tverberg Conjecture then states that there is no almost r-immersion from $\Delta^{(d+1)(r-1)}$ to $\mathbb{R}^d$. We hope that considering the manifold calculus of functors applied to the space of almost r-immersions of $\Delta^{(d+1)(r-1)}$ into $\mathbb{R}^d$ will yield information about the Topological Tverberg Conjecture.

In this thesis, we begin the study of almost r-immersions by studying the related space of *r-immersions*. An r-immersion is a map f from a manifold M to a manifold N (in our case, $N = \mathbb{R}^d$) such that f has no r-fold self-intersections. The space of r-immersions is easier to consider than the space of almost r-immersions: for an r-immersion, the definition forbids all r-fold intersection points, while for an almost r-immersion, r-fold intersection points are allowed as long the preimages come from faces that are not pairwise disjoint. Furthermore, the Taylor tower for embeddings is known, and has been described in [**13**], [**6**], and [**9**] (among others).

Since the space of 2-immersions is the same as the space of embeddings, the two spaces have the same Taylor tower. We hypothesize that the Taylor tower for r-immersions when $r > 2$ is a "spread out" version of the Taylor tower for embeddings. In particular, we believe that the kth stage of the Taylor tower for r-immersions is precisely the $\left(\lfloor \frac{k}{r-1} \rfloor + 1\right)$th stage of the Taylor tower for embeddings. While we do not prove this description of the Taylor tower in this thesis, we do prove that the first $r-1$ stages of the Taylor tower for r-immersions are all equivalent to the first stage of the tower.

What follows is an outline of this thesis.

Chapter 1 contains topological background including definitions and examples that are needed for the remaining sections. In particular, we define the spaces of embeddings and immersions, discuss fibrations and homotopy fibers, and define the connectivity of a space and of a map.

Chapter 2 gives definitions directly related to the Topological Tverberg Conjecture, and two different formulations of the conjecture. In addition, we present an outline of the work that produced counterexamples to the conjecture using a higher-dimensional Whitney Trick.

Chapter 3 discusses the manifold calculus of functors as applied to the space of embeddings. We outline an argument for the convergence of the Taylor tower of embeddings.

Chapter 4 presents our work on the manifold calculus of functors applied to r-immersions. We begin the chapter with a discussion of why we believe that the manifold calculus of functors should yield solutions to the Topological Tverberg Conjecture. We conclude this thesis by presenting our results on the manifold calculus of functors applied to r-immersions, as well as remarks on potential future work.

CHAPTER 1

Topological Background

This section contains background from algebraic topology that will be needed for our discussion of the Topological Tverberg Conjecture and the manifold calculus of functors.

1.1. Spaces of Embeddings and Immersions

Throughout this section, let M and N be smooth manifolds. For the Topological Tverberg Conjecture, we will want to eventually consider simplicial complexes. We discuss smooth manifolds here, since the results for the manifold calculus of functors that we present are for smooth manifolds. The manifold calculus of functors can be adapted to simplicial complexes as by Tillman in [**12**].

DEFINITION 1.1.1. An *immersion* of M in N is a smooth map $f : M \hookrightarrow N$ whose derivative is injective. The *space of immersions of M in N* is denoted $\mathrm{Imm}(M, N)$ (with the weak $\mathcal{C}^\infty$ topology).

EXAMPLE 1.1.2. Let $*$ be a point. Then $\mathrm{Imm}(*, \mathbb{R}^d) = \mathbb{R}^d \simeq *$. Now let $\sqcup_r *$ be the disjoint union of r points. Then $\mathrm{Imm}(\sqcup_r *, \mathbb{R}^d) = (\mathbb{R}^d)^r \simeq *$.

DEFINITION 1.1.3. An *r-immersion* of M in N is a smooth map $f : M \hookrightarrow N$ that has no r-fold self-intersections. That is, for any $x_1, \ldots, x_r$, not all of $f(x_1), \ldots, f(x_r)$ are equal. The *space of r-immersions of M in N* is denoted $r\,\mathrm{Imm}(M, N)$ (with the weak $\mathcal{C}^\infty$ topology).

This definition allows up to r-fold self-intersections of maps. However, if an immersion has no self-intersections, it is actually an embedding.

DEFINITION 1.1.4. A *(smooth) embedding* of M in N is a smooth injective map $f : M \hookrightarrow N$ whose derivative is injective and that is a homeomorphism onto its image where the image has the subspace topology from N. The *space of embeddings of M in N* is denoted $\mathrm{Emb}(M, N)$ (with the weak $\mathcal{C}^\infty$ topology).

EXAMPLE 1.1.5. The space of embeddings $\mathrm{Emb}(*, \mathbb{R}^d)$ equals $\mathbb{R}^d \simeq *$, just as in the case for immersions. However, the space $\mathrm{Emb}(\sqcup_r *, \mathbb{R}^d) = \{(x_1, \ldots, x_r) \in (\mathbb{R}^d)^r \mid x_i \neq x_j \text{ for } i \neq j\}$, which is not contractible.

The second space in Example 1.1.5 is an important space for our work on the Topological Tverberg Conjecture.

DEFINITION 1.1.6. Let X be a topological space, and $n \in \mathbb{Z}_{\geq 1}$. Then the *configuration space of n points in X* is defined as

$$\mathrm{Conf}(n, X) := \{(x_1, \ldots, x_n) \in X^n \mid x_i \neq x_j \text{ for } i \neq j\}.$$

Note that $\mathrm{Conf}(n, X)$ is the same as $\mathrm{Emb}(\{x_1, \ldots, x_n\}, X)$. The definition of configuration spaces forbids two-fold intersections, but for the Tverberg conjecture we are interested in r-fold intersections.

REMARK 1.1.7. $\mathrm{Conf}(n, X)$ is the space of *ordered* configurations of n points in X. There is a natural action of Σ_n on $\mathrm{Conf}(n, X)$, and when we quotient $\mathrm{Conf}(n, X)$ by the action of Σ_n, we obtain the *unordered* configuration space of n points in X, denoted $\binom{X}{n}$.

DEFINITION 1.1.8. The *r-configuration space of n points in X* is

$$r\,\mathrm{Conf}(n, X) := \{(x_1, \ldots, x_n) \in X^n \mid \text{ no } r \text{ of the } x_i \text{ are equal}\}.$$

Note that $r\,\mathrm{Conf}(n, X)$ is the same as $r\,\mathrm{Imm}(\{x_1, \ldots, x_n\}, X)$. The space $r\,\mathrm{Conf}(n, X)$ is closer to what we will want to work with for the Topological Tverberg Conjecture.

EXAMPLE 1.1.9. Let $m < r$, and consider $r\,\mathrm{Conf}(m, \mathbb{R}^d)$. Since $m < r$, we are looking at m-tuples $\{x_1, \ldots, x_m\}$ of points in $\mathbb{R}^d$ where any number of the x_i may be equal. Therefore, $r\,\mathrm{Conf}(m, \mathbb{R}^d) = (\mathbb{R}^d)^m \simeq *$.

1.2. Fibrations and Homotopy Fibers

DEFINITION 1.2.1. Let X and Y be topological spaces, and $y \in Y$. For a map $f : X \to Y$, the space $F_y = f^{-1}(y)$ is called the *fiber of f over y*.

Often, the fiber of a map depends on choice of y.

EXAMPLE 1.2.2. Consider the projection map

$$\pi : 3\,\mathrm{Conf}(3, \mathbb{R}^2) \to 3\,\mathrm{Conf}(2, \mathbb{R}^2)$$

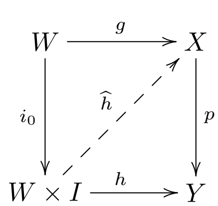
given by $\pi((x_1, x_2, x_3)) = (x_1, x_2)$. Then π is surjective, since there are no restrictions on a point $(x_1, x_2) \in 2\,\mathrm{Conf}(2, \mathbb{R}^2)$.

If $x_1 \neq x_2$, then taking the fiber of π over (x_1, x_2) yields $\mathbb{R}^2 \simeq *$, since x_3 can equal any point in $\mathbb{R}^2$ (including either x_1 or x_2).

However, if $x_1 = x_2 = a$ for some $a \in \mathbb{R}^2$, then taking the fiber over (x_1, x_2) yields $\mathbb{R}^2 - \{a\} \simeq S^1$, since x_3 can be any point in $\mathbb{R}^2$ except for a.

We are interested in defining a type of map that excludes the previous example, that is, such that all fibers are homotopy equivalent.

DEFINITION 1.2.3. A map $f : X \to Y$ is a *fibration* if for all spaces W and commutative squares



a map $\widehat{h}$ exists that makes the diagram commute. In the diagram, $i_0 : W \to W \times I$ is given by $w \mapsto (w, 0)$.

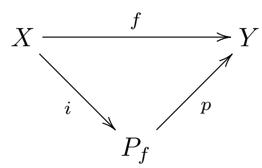
PROPOSITION 1.2.4. *If $p : X \to Y$ is a fibration, then the fibers $F_y = p^{-1}(y)$ are homotopy equivalent for all y in the same path component of Y.*

Example 1.2.2 gives an example of a map that is not a fibration, since the fibers taken over different values of y are not homotopy equivalent. For any map $f : X \to Y$, we can replace X by a "thicker" space that is homotopy equivalent to X, and such that the fibers of the induced map become homotopy equivalent.

Definition 1.2.5. Let $f : X \to Y$ be a map. Then the *mapping path space of* f, denoted P_f, is the subspace of $X \times \mathrm{Map}(I, Y)$ given by

$$P_f = \{(x, \alpha) \mid x \in X, \alpha : I \to Y, \alpha(1) = f(x)\}.$$

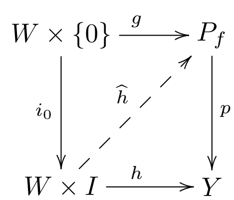
Let $c_y \in \mathrm{Map}(I, Y)$ be the constant map at y. Let i be the inclusion $X \hookrightarrow P_f$ such that $i(x) = (x, c_{f(x)})$, and define $p : P_f \to Y$ by $p(x, \alpha) = \alpha(1)$. Then f factors through P_f as



Note that $P_f \simeq X$. If we replace X by P_f and f by p, we obtain a fibration $p : P_f \to Y$.

Proposition 1.2.6. *The evaluation map* $p : P_f \to Y$ *is a fibration.*

Proof. For a space W, consider a commutative square



We must produce a map $\widehat{h}$ that makes the diagram commute. Write $g(w) = (x_w, \gamma_w)$. Define $\widehat{h} : W \times I \to P_f$ by

$$\widehat{h}(w, s) = (x_w, \alpha_{(w,s)})$$

where

$$\alpha_{(w,s)}(t) = \begin{cases} \gamma_w(t + ts) & \text{if } 0 \le t \le \frac{1}{1+s} \\ h(w, t + st - 1) & \text{if } \frac{1}{1+s} \le t \le 1. \end{cases}$$

Then

$$\widehat{h} \circ i_0((w, 0)) = (x_w, \alpha_{(w,0)}) = (x_w, \gamma_w) = g(w),$$

and

$$p \circ \widehat{h}(w, s) = \alpha_{((w,s))}(1) = h(w, s)$$

as desired. □

Definition 1.2.7. Given a map $f : X \to Y$, the *homotopy fiber of* f *over* $y \in Y$, denoted $\mathrm{hofiber}_y(f)$ is the subspace of $X \times \mathrm{Map}(I, Y)$ given by

$$\mathrm{hofiber}_y(f) := \{(x, \alpha) \mid x \in X,\ \alpha : I \to Y,\ \alpha(0) = f(x),\ \alpha(1) = y\}.$$

The homotopy fiber is invariant under homotopy equivalence. That is, when X or Y is replaced by a space that is homotopy equivalent to the original space, the homotopy fiber of the map remains the same up to homotopy.

Example 1.2.8. Let X and Y be contractible spaces, and consider a map $f : X \longrightarrow Y$. Since X and Y are both contractible and the homotopy fiber is invariant under homotopy, we can consider instead $f : * \longrightarrow *$, which is a fibration with fiber $*$. Therefore, the homotopy fiber of f is contractible.

Example 1.2.9. Let $f : X \longrightarrow Y$, where X is a topological space and Y is contractible. An element of the homotopy fiber over a point $y \in Y$ is given by a point $x \in X$ together with a path α in Y from $f(x)$ to y. Since homotopy fibers are invariant under homotopy and $Y \simeq *$, we can replace Y by the single point space $*$ and consider the constant map

$$X \longrightarrow *.$$

The only path in the single point space is the constant path. Denote this constant path by α_*. Then the homotopy fiber is given by

$$\{(x, \alpha_*) \mid x \in X\} \simeq X.$$

The homotopy fiber of f is therefore homotopy equivalent to X.

Example 1.2.10. Recall the projection map

$$\pi : 3\,\mathrm{Conf}(3, \mathbb{R}^2) \longrightarrow 3\,\mathrm{Conf}(2, \mathbb{R}^2)$$

given by $\pi((x_1, x_2, x_3)) = (x_1, x_2)$. Since $3\,\mathrm{Conf}(2, \mathbb{R}^2) = (\mathbb{R}^2)^2 \simeq *$, the previous example tells us that $\mathrm{hofiber}\,\pi \simeq 3\,\mathrm{Conf}(3, \mathbb{R}^2)$.

In general, when

$$\pi : r\,\mathrm{Conf}(r, \mathbb{R}^d) \longrightarrow r\,\mathrm{Conf}(r-1, \mathbb{R}^d)$$

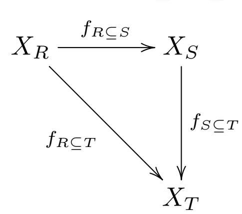
is projection onto any $r-1$ coordinates, the homotopy fiber of π is homotopy equivalent to $r\,\mathrm{Conf}(r, \mathbb{R}^d)$. In cases where we have

$$\pi : r\,\mathrm{Conf}(m, \mathbb{R}^d) \longrightarrow r\,\mathrm{Conf}(m-1, \mathbb{R}^d)$$

such that $m < r$, the homotopy fiber is contractible, since both spaces are contractible.

For an integer $n \geq 0$, let $\underline{n} = \emptyset$ if $n = 0$ and $\underline{n} = \{1, \ldots, n\}$ otherwise.

Definition 1.2.11. An n-cube $\mathcal{X}$ is a diagram of spaces consisting of a space $\mathcal{X}(S) = X_S$ for each $S \subseteq \underline{n}$, and a map $\mathcal{X}(S \subseteq T) = f_{S \subseteq T} : X_S \to X_T$ for each $S \subseteq T$ such that $f_{S \subseteq S} = 1_{X_S}$ and for all $R \subseteq S \subseteq T$, the diagram



commutes.

Definition 1.2.12. Let $\mathcal{X}$ be an n-cube of based spaces. Then the *total (homotopy) fiber of* $\mathcal{X}$, denoted $\mathrm{tfiber}(\mathcal{X})$ is defined iteratively as

- If $\underline{n} = \emptyset$, then $\mathrm{tfiber}(\mathcal{X}) = X_\emptyset$.
- If $\underline{n} \neq \emptyset$, view $\mathcal{X} = \mathcal{Y} \to \mathcal{Z}$ as a map of $(n-1)$-cubes, and then

$$\mathrm{tfiber}(\mathcal{X}) \simeq \mathrm{hofiber}(\mathrm{tfiber}(\mathcal{Y}) \to \mathrm{tfiber}(\mathcal{Z})).$$

This definition does not depend on how one views $\mathcal{X}$ as a map of $(n-1)$-cubes (see [**10**] for details).

One would calculate the total homotopy fiber of the square $\mathcal{X}$ given by

$$\begin{array}{ccc} X_\emptyset & \longrightarrow & X_1 \\ \downarrow & & \downarrow \\ X_2 & \longrightarrow & X_{12} \end{array}$$

either by first taking homotopy fibers vertically and then taking the homotopy fiber of the resulting (horizontal) map, or by first taking the homotopy fibers horizontally and then taking the homotopy fiber of the resulting vertical map. The first option is shown in the diagram below.

$$\begin{array}{ccccc} \mathrm{tfiber}(\mathcal{X}) & \dashrightarrow & \mathrm{hofiber}(X_\emptyset \to X_2) & \dashrightarrow & \mathrm{hofiber}(X_1 \to X_{12}) \\ & & \downarrow & & \downarrow \\ & & X_\emptyset & \longrightarrow & X_1 \\ & & \downarrow & & \downarrow \\ & & X_2 & \longrightarrow & X_{12} \end{array}$$

where

$$\mathrm{tfiber}(\mathcal{X}) = \mathrm{hofiber}(\mathrm{hofiber}(X_\emptyset \to X_2) \longrightarrow \mathrm{hofiber}(X_1 \to X_{12})).$$

EXAMPLE 1.2.13. If the square in the above diagram is

$$\begin{array}{ccc} \mathrm{Conf}\left(2, \mathbb{R}^d\right) & \longrightarrow & \mathrm{Conf}\left(1, \mathbb{R}^d\right) \\ \downarrow & & \downarrow \\ \mathrm{Conf}\left(1, \mathbb{R}^d\right) & \longrightarrow & \mathrm{Conf}\left(\emptyset, \mathbb{R}^d\right) \end{array}$$

then the fiber of the left vertical map over some point $y \in \mathrm{Conf}\left(1, \mathbb{R}^d\right) \simeq \mathbb{R}^d$ and of the right vertical map over $*$ gives the inclusion map

$$i : \left(\mathbb{R}^d - \{y\}\right) \longrightarrow \mathbb{R}^d,$$

which has homotopy fiber $\mathbb{R}^d - \{y\} \simeq S^{d-1}$. Therefore, the total homotopy fiber of the given square is homotopy equivalent to S^{d-1}.

1.3. Connectivities of Spaces and Maps

DEFINITION 1.3.1. A *space X is k-connected* if, for all $0 \le i < (k+1)$, every map $S^i \to X$ extends to a map $D^{i+1} \to X$.

PROPOSITION 1.3.2. *The following are equivalent for $k \in \mathbb{Z} \geq 0$:*

(1) *X is k-connected.*
(2) *For all $0 \le i \le k$ and basepoints x_0 of X, we have $\pi_i(X, x_0) = 0$.*

EXAMPLE 1.3.3.

- $\mathbb{R}^d$ is ∞-connected for any $d \in \mathbb{Z}_{\geq 0}$.

- The n-sphere S^n is $(n-1)$-connected.

DEFINITION 1.3.4. A *map* $f : X \to Y$ *is* k*-connected* if its homotopy fiber is $(k-1)$-connected.

This definition is equivalent to the usual definition of a k-connected map, which says that a map $f : X \to Y$ is k-connected if it induces an isomorphism on π_i for $i \leq (k-1)$ and a surjection on π_k.

EXAMPLE 1.3.5. Recall from Example 1.2.13 that the homotopy fiber of the inclusion map

$$(\mathbb{R}^d - \{y\}) \longrightarrow \mathbb{R}^d$$

is $(\mathbb{R}^d - \{y\}) \simeq S^{d-1}$. This is a $(d-2)$-connected space. Therefore, the map $(\mathbb{R}^d - \{y\}) \longrightarrow \mathbb{R}^d$ is a $((d-2)+1) = (d-1)$-connected map.

Ultimately, we will be interested in looking at connectivities of total homotopy fibers of cubes.

CHAPTER 2

The Topological Tverberg Conjecture

Recent work on the Topological Tverberg Conjecture has used tools from combinatorial topology. However, this thesis will explore a new approach that we hope will be applicable to other combinatorial topology problems. We begin this section with a few basic definitions, and continue to describe some of the more recent work on the conjecture.

2.1. Background and Statement

DEFINITION 2.1.1. An *N-dimensional simplex* (or N-simplex), denoted Δ^N, is the intersection of all convex sets containing some fixed $N+1$ affinely independent points in $\mathbb{R}^n$ for some $n > N$.

For example, a 0-simplex is a point, a 1-simplex is a line segment joining two points, a 2-simplex is a triangle, and a 3-simplex is a tetrahedron.

DEFINITION 2.1.2. A *face of an N-simplex* is a k-dimensional simplex whose vertex set is a subset of the vertex set of Δ^N.

Thus, faces are closed in the N-simplex. For a set of r faces to be pairwise disjoint, no two of the faces can share a vertex. The conjecture is as follows.

CONJECTURE 2.1.3 (Topological Tverberg Conjecture). *For integers $r \geq 2$ and $d \geq 1$, let $N := (d+1)(r-1)$. Then for every continuous $f : \Delta^N \to \mathbb{R}^d$, there exist pairwise-disjoint faces $\sigma_1, \sigma_2, \ldots, \sigma_r$ of Δ^N such that $f(\sigma_1) \cap f(\sigma_2) \cap \ldots \cap f(\sigma_r)$ is non-empty.*

Conjecture 2.1.3 is true in the case where r is a power of a prime. Until recently, the conjecture was believed to hold for all integers $r \geq 2$ (e.g. [**7**]). However, a counterexample was announced by Florian Frick in February of 2015 [**4**], and other counterexamples were found later by Avvakumov, Mabillard, Skopenkov, and Wagner in [**1**]. Section 2.2 summarizes the argument in [**4**] and introduces some definitions that will be necessary later in the paper. We follow [**11**], which is a "user's guide" to the proof of Frick's counterexample, and contains more detail than the original paper.

DEFINITION 2.1.4. K is a *simplicial complex*, if it is a set of simplices such that

- any face of a simplex in K is also in K, and
- the intersection of any two faces Δ^m and Δ^ℓ of K is either $\emptyset$ or a face of both Δ^m and Δ^ℓ.

DEFINITION 2.1.5. Let K be a simplicial complex. A continuous map $f : K \to \mathbb{R}^d$ is an *almost r-immersion* if $f(\sigma_1) \cap \ldots f(\sigma_r) = \emptyset$ for any $\sigma_1, \ldots, \sigma_r$ pairwise-disjoint faces of K.

REMARK 2.1.6. In the combinatorial topology literature such as [**4**] and [**11**], the maps in Definition 2.1.5 are called *almost r-embeddings*. Since the definition of embeddings forbids any self-intersections at all (see Definition 1.1.4), we have changed the terminology to immersions.

2.2. Re-Statement and Current Work

Using Definition 2.1.5, we can restate the conjecture.

CONJECTURE 2.2.1 (Topological Tverberg Conjecture). *For integers $r \geq 2$ and $d \geq 1$, there does not exist an almost r-immersion $f : \Delta^{(d+1)(r-1)} \to \mathbb{R}^d$.*

As noted above, Conjecture 2.2.1 holds when r is a power of a prime. The following theorem gives a condition on d for which the conjecture fails.

THEOREM 2.2.2 ([**4**], [**11**]). *If r is not a prime power and $d \geq 3r + 1$, then there is an almost r-immersion of $\Delta^{(d+1)(r-1)}$ in $\mathbb{R}^d$.*

EXAMPLE 2.2.3. By Theorem 2.2.2, there is an almost 6-immersion of Δ^{100} in $\mathbb{R}^{19}$.

Counterexamples for lower values of d are given in [**1**], and the problem is still open for $d < 12$.

The proof of the theorem is outside the scope of this thesis, so we will not present the entire proof. Below is a basic outline of the argument, which requires the definition of a $\mathbb{Z}$-almost r-immersion. Since this definition is not required for our work, we present it without explanation. For more details, the reader can see Section 3 of [**11**].

DEFINITION 2.2.4. Let $k, r \in \mathbb{Z}$ with $k \geq 1$ and $r \geq 2$, and $\sigma_1, \ldots, \sigma_r$ be pairwise disjoint faces of a $k(r-1)$-dimensional simplicial complex K. Suppose also that $f : K \to \mathbb{R}^{kr}$ is a piecewise-linear map in general position. Then f is a *$\mathbb{Z}$-almost r-immersion* if the sum of the r-intersection signs of all global r-fold points $y \in f(\sigma_1) \cap \ldots \cap f(\sigma_r)$ is zero.

We discuss the case where $d = 3r + 1$. Consider the following three statements:

(1) There is a $\mathbb{Z}$-almost r-immersion of each $3(r-1)$-complex in $\mathbb{R}^{3r}$.
(2) There is an almost r-immersion of each $3(r-1)$-complex in $\mathbb{R}^{3r}$.
(3) There is an almost r-immersion of the $(d+1)(r-1)$-simplex in $\mathbb{R}^d$, where $d = 3r + 1$.

Then a counterexample to Conjecture 2.2.1 follows from showing that Statement 1 holds, that Statement 1 implies Statement 2, and that Statement 2 implies Statement 3. Part of the proof involves a higher-dimensional Whitney trick. This higher-dimensional Whitney trick was developed in [**13**] to remove r-fold intersections from functions mapping N-simplices to $\mathbb{R}^d$.

The original Whitney trick, which removes 2-fold intersections from maps under certain conditions, can be recovered using a technique called manifold calculus of functors. Since we are interested in removing r-fold intersections from maps, we believe that the manifold calculus of functors is related to the Topological Tverberg Conjecture. The remainder of this paper is devoted to the manifold calculus of functors, first as applied to embeddings, and then as applied to the Topological Tverberg Conjecture.

CHAPTER 3

Manifold Calculus of Functors

We begin this chapter with a few definitions to introduce the manifold calculus of functors, and then proceed with the case of embeddings. The following chapter will relate the work in this chapter to the Topological Tverberg Conjecture.

3.1. Category Theory Background

DEFINITION 3.1.1. A *category* $\mathcal{C}$ consists of

- a class of objects, denoted $\mathrm{Ob}(\mathcal{C})$,
- a class of morphisms between objects, denoted $\hom(\mathcal{C})$, such that each object $X \in \mathrm{Ob}(\mathcal{C})$ has an identity morphism id_X, and composition of morphisms is associative.

EXAMPLE 3.1.2.

- The category **Set** has sets as the objects, and functions between sets as morphisms (with the usual composition).
- The category of groups has groups as objects, and group homomorphisms as morphisms.
- The category **Top** has topological spaces as objects, and continuous functions as morphisms.

DEFINITION 3.1.3. The dual category (or opposite category) for a given category $\mathcal{C}$ is the category with objects $\mathrm{Ob}(\mathcal{C})$ but with the original morphisms reversed.

DEFINITION 3.1.4. A *(covariant) functor* F is a map between categories $\mathcal{C}$ and $\mathcal{D}$. The functor F associates to each object X in $\mathcal{C}$ an object $F(X)$ in $\mathcal{D}$. Furthermore, F associates to each morphism $\alpha : X \to Y$ in $\hom(\mathcal{C})$ a morphism $F(\alpha) : F(X) \to F(Y)$ in $\hom(\mathcal{D})$ such that

- for all $X \in \mathrm{Ob}(\mathcal{C})$ we have $F(\mathrm{id}_X) = id_{F(X)}$, and
- $F(\alpha \circ \beta) = F(\alpha) \circ F(\beta)$ for $\alpha : Y \to Z$ and $\beta : X \to Y$ where $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$.

DEFINITION 3.1.5. A functor F is called *contravariant* if it reverses the direction of the morphisms. That is, F is contravariant if each morphism $\alpha : X \to Y$ gets mapped to a morphism $F(\alpha) : F(Y) \to F(X)$.

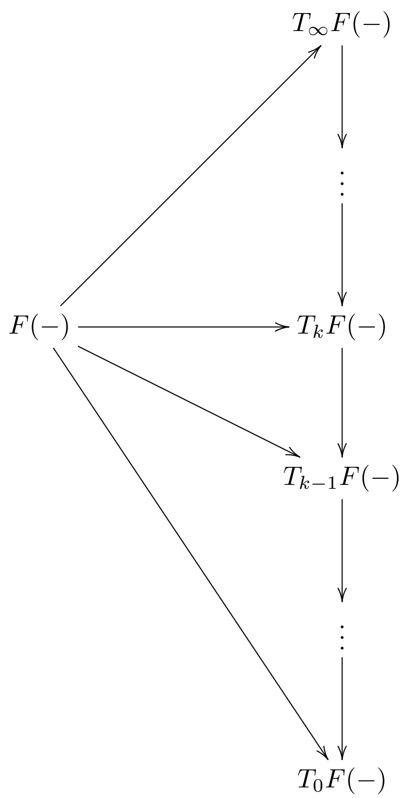
EXAMPLE 3.1.6. There is a functor from the category of groups to the category of sets, which maps each group to its underlying set (i.e. “forgets” about the group structure). This functor is covariant.

EXAMPLE 3.1.7. The powerset functor $P : \mathbf{Set} \longrightarrow \mathbf{Set}$ sends each set X to its power set $\mathcal{P}(X)$. Each function $f : X \longrightarrow Y$ is sent to the map that sends each $U \subseteq X$ to its image $f(U) \subseteq Y$. The functor P is covariant.

There is also a contravariant powerset functor $P^{op} : \mathbf{Set} \longrightarrow \mathbf{Set}$. Like P, the functor P^{op} also sends each set X to its powerset. However, each map $f : X \longrightarrow Y$ is sent to the map that sends each $V \subseteq Y$ to its preimage $f^{-1}(V) \subseteq X$.

3.2. General Setup for Manifold Calculus of Functors

The calculus of functors is a technique that allows us to approximate functors. The technique is analogous to the Taylor expansion for real-valued functions, which approximates differentiable functions via polynomials. In particular, for each functor $F : \mathcal{C} \to \mathcal{D}$, we obtain a Taylor tower of functors and spaces with natural transformations between them as shown in the following diagram:



Here, $T_\infty F(-)$ is the inverse limit of the tower. We say that the tower converges if the map $F(-) \to T_\infty F(-)$ is an equivalence, that is, if the map is infinitely connected.

Let **Top** be the category of topological spaces, and let M be a manifold. Take $\mathcal{O}(M)$ to be the category of open subsets of M, with inclusions as the morphisms. The manifold calculus of functors studies functors

$$F : \mathcal{O}(M)^{op} \to \mathbf{Top}$$

where "*op*" means the opposite category (see Definition 3.1.3). Let $\mathcal{O}_k(-)$ be the subcategory of $\mathcal{O}(-)$ consisting of open subsets of M diffeomorphic to up to k disjoint balls. Then for $U \subset M$, we define the kth stage of the Taylor tower as

$$T_k F(U) = \operatorname*{holim}_{V \in \mathcal{O}_k(U)} F(U),$$

where holim stands for homotopy limit. For our purposes, we do not need the definition of homotopy limit, as there are alternate ways of computing the kth stage of the Taylor tower. Technical details can be found in [**10**]. The homotopy limit, which is similar to the limit of a diagram except that it is homotopy invariant, is difficult to work with, but it can be thought of as trying to reconstruct $F(U)$ from the open balls V in $\mathcal{O}_k(U)$. However, we can also find T_kF using cubical diagrams (at the expense of some functoriality properties).

3.3. The Embedding Functor

In general, one can consider embeddings of M in some other manifold N, but since we will eventually be applying the manifold calculus of functors to maps taking simplices to $\mathbb{R}^d$, we consider the case $N = \mathbb{R}^d$.

The embedding space $\operatorname{Emb}(M, \mathbb{R}^d)$ can be thought of as a functor

$$\operatorname{Emb}(M, \mathbb{R}^d) : \mathcal{O}(M)^{op} \to \mathbf{Top}\,.$$

so that for each inclusion

$$U_1 \hookrightarrow U_2$$

of open subsets of M, there is a restriction

$$\operatorname{Emb}(U_2, \mathbb{R}^d) \longrightarrow \operatorname{Emb}(U_1, \mathbb{R}^d).$$

We are interested in the Taylor tower of the space r-immersions thought of as a functor (and eventually, the space of almost r-immersions, see Definition 2.1.5). We believe that the Taylor tower of the space of r-immersions is related to that of embeddings. Therefore, we first present the case for embeddings.

Note that $\operatorname{Emb}(M, \mathbb{R}^n) \subset \operatorname{Imm}(M, \mathbb{R}^n)$, so there is an inclusion

$$i : \operatorname{Emb}(M, \mathbb{R}^n) \hookrightarrow \operatorname{Imm}(M, \mathbb{R}^n).$$

This observation, in conjunction with several results in manifold calculus, leads us to the following theorem, from [**13**].

Theorem 3.3.1. *The first stage of the Taylor tower for the functor* $\operatorname{Emb}(M, \mathbb{R}^d)$ *is equivalent to immersions of* M *in* $\mathbb{R}^d$*. That is,*

$$T_1 \operatorname{Emb}(M, \mathbb{R}^d) \simeq \operatorname{Imm}(M, \mathbb{R}^d).$$

The above theorem tells us that the first approximation of the space of embeddings is the space of immersions.

For the Taylor tower for the $\operatorname{Emb}(M, \mathbb{R}^d)$ functor to converge means that the map

$$\operatorname{Emb}(M, \mathbb{R}^d) \longrightarrow T_\infty \operatorname{Emb}(M, \mathbb{R}^d)$$

is infinitely connected. If we can establish that the connectivity of the map

$$\operatorname{Emb}(M, \mathbb{R}^d) \longrightarrow T_k \operatorname{Emb}(M, \mathbb{R}^d)$$

tends to infinity as k goes to infinity, then this would show that the tower for Emb converges. This turns out to be the case, under certain conditions on the dimension of M.

Theorem 3.3.2 ([**5**]). *Let m be the dimension of M. Then the map*

$$\operatorname{Emb}(M, \mathbb{R}^d) \longrightarrow T_k \operatorname{Emb}(M, \mathbb{R}^d)$$

is $(1 - m + k(d - m - 2))$*-connected.*

As desired, this theorem establishes that the connectivity of the map

$$\mathrm{Emb}(M,\mathbb{R}^d) \longrightarrow T_k\,\mathrm{Emb}(M,\mathbb{R}^d)$$

grows with k (as long as $d > (m+2)$). What follows is an outline of the proof of Theorem 3.3.2. We would like to imitate the argument to obtain a similar result for the connectivity of the maps

$$r\,\mathrm{Imm}(M,\mathbb{R}^d) \longrightarrow T_k r\,\mathrm{Imm}(M,\mathbb{R}^d).$$

The connectivity of the map $\mathrm{Emb}(M,\mathbb{R}^d) \to T_k\,\mathrm{Emb}(M,\mathbb{R}^d)$ is related to the connectivity of the map $T_k\,\mathrm{Emb}(M,\mathbb{R}^d) \to T_{k-1}\,\mathrm{Emb}(M,\mathbb{R}^d)$. If we know the connectivity of

$$\mathrm{Emb}(M,\mathbb{R}^d) \longrightarrow T_{k-1}\,\mathrm{Emb}(M,\mathbb{R}^d)$$

and of

$$T_k\,\mathrm{Emb}(M,\mathbb{R}^d) \longrightarrow T_{k-1}\,\mathrm{Emb}(M,\mathbb{R}^d),$$

then we will be able find a lower bound for the connectivity of

$$\mathrm{Emb}(M,\mathbb{R}^d) \longrightarrow T_k\,\mathrm{Emb}(M,\mathbb{R}^d).$$

Therefore, we only need to know the connectivity of

$$\mathrm{Emb}(M,\mathbb{R}^d) \longrightarrow T_1\,\mathrm{Emb}(M,\mathbb{R}^d)$$

and the connectivity of

$$T_k\,\mathrm{Emb}(M,\mathbb{R}^d) \longrightarrow T_{k-1}\,\mathrm{Emb}(M,\mathbb{R}^d)$$

in terms of k.

In turn, knowing the connectivity of $T_k\,\mathrm{Emb}(M,\mathbb{R}^d) \to T_{k-1}\,\mathrm{Emb}(M,\mathbb{R}^d)$ is the same as knowing the connectivity of its homotopy fiber.

Definition 3.3.3. The *kth layer of the Taylor tower* of a functor $F(-)$ is defined as

$$L_kF(-) := \mathrm{hofiber}(T_kF(-) \longrightarrow T_{k-1}F(-)).$$

This is also called a *homogeneous degree k polynomial functor*, and a theorem in [**13**] gives us a description of $L_kF(-)$. See also Theorem 10.2.23 in [**10**] for details.

Theorem 3.3.4. *Let Γ^c denote the space of compactly supported sections of a map. Let M be a smooth manifold with $U \subset M$, and $\binom{U}{k}$ be the unordered configuration space of k points in U. Then every homogenous degree k polynomial functor is weakly equivalent to the space of sections of a fibration over $\binom{M}{k}$, i.e., there is a fibration*

$$\Phi : Z \longrightarrow \binom{M}{k}$$

for some space Z such that the functor

$$L_kF(U) \simeq \Gamma^c\left(\binom{U}{k}, Z; \Phi\right)$$

for all $U \subset M$.

We omit the proof. For our work, we do not need to worry about the definition of compactly supported sections. In [**13**], M is a compact manifold with boundary. Since we are interested in $M = \Delta^N$, and Δ^N is compact with boundary, we can apply Theorem 3.3.4.

The question of the connectivity of the map $F(-) \to T_k F(-)$ then comes down to the connectivity of the space of sections in Theorem 3.3.4. The following proposition tells us that the connectivity of the space of sections is just the connectivity of the fiber of Φ minus the dimension of the base $\binom{M}{k}$ of Φ. The dimension of $\binom{M}{k}$ is $k \cdot \dim(M)$.

PROPOSITION 3.3.5 (Prop. 10.2.26, [**10**]). *Let $m = \dim M$. For a contravariant functor F, if the fiber of Φ is c_k-connected, then $L_k F(M)$ is $(c_k - km)$-connected.*

In Proposition 3.3.5, F has to satisfy some more technical conditions; the functors we will be using satisfy these conditions.

Since Φ is a fibration, the fiber over an arbitrary point $S \in \binom{M}{k}$ is homotopy equivalent to the fiber over any other point.

THEOREM 3.3.6 (Theorem 5.6 in [**9**]). *The fiber Φ_S, is the total homotopy fiber of a k-cube of spaces made up of values of F on tubular neighborhoods of S.*

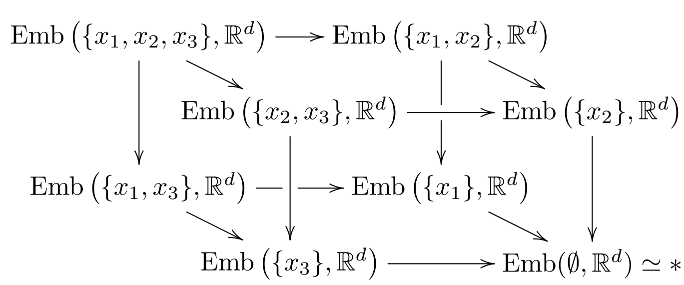
If $F = \mathrm{Emb}(-, \mathbb{R}^d)$, then Theorem 3.3.6 says that we can also think of Φ_S as the total homotopy fiber of a cubical diagram of embeddings of disks. Since disks are contractible, the space of embeddings of some number of disks is equivalent to the space of embeddings of the same number of points. Therefore, the space of embeddings of k disks in $\mathbb{R}^d$ is equivalent to $\mathrm{Emb}\big(\{x_1, \dots, x_k\}, \mathbb{R}^d\big)$, which is $\mathrm{Conf}(k, \mathbb{R}^d)$.

The connectivity of the total homotopy fiber in Theorem 3.3.6 gives us the value of

$$\text{connectivity of } \Phi_S = c_k,$$

and then Proposition 3.3.5 gives us the connectivity of $L_k \,\mathrm{Emb}(M, \mathbb{R}^d)$.

EXAMPLE 3.3.7. If $k = 3$, then Φ_S is the total homotopy fiber of the cube



Therefore, the problem of the connectivity of the layers of the Taylor tower for the functor $\mathrm{Emb}(M, \mathbb{R}^n)$ comes down to the connectivity of the homotopy fibers of projection maps between configuration spaces. The projection maps turn out to be fibrations ([**3**]), which means that the homotopy fiber is just the fiber. The fiber of a projection map $\mathrm{Conf}(k, \mathbb{R}^d) \to \mathrm{Conf}(k-1, \mathbb{R}^d)$ is the space of all of the places that a kth point could be placed in $\mathbb{R}^d$, given $k-1$ points where it cannot be placed. That is, the fiber of $\mathrm{Conf}(k, \mathbb{R}^d) \to \mathrm{Conf}(k-1, \mathbb{R}^d)$ is exactly $\mathbb{R}^d$ with $k-1$ points removed, which is homotopy equivalent to a wedge of spheres.

For the cube in Example 3.3.7, it turns out (Example 5.5.6, [**10**]) that the total fiber is $\Omega S^{d-1} * \Omega S^{d-1}$, where ΩS^{d-1} is the loop space of S^{d-1} and $*$ means the join of the two spaces. Since ΩS^{d-1} is $(d-3)$-connected and the connectivity of the join of two spaces is 2 greater than the sum of the connectivities, we have that $\Omega S^{d-1} * \Omega S^{d-1}$ is $2(d-3)+2 = (2d-4)$-connected. Then using Proposition 3.3.5, we have that

$$\text{connectivity of } L_3 \operatorname{Emb}(M, \mathbb{R}^d) = (2d-4) - 3 \cdot \dim(M).$$

In the following proposition, the cube generated by $\operatorname{Emb}\big(\{x_1, \dots, x_k\}, \mathbb{R}^d\big)$ for some k is the k-cube formed by first projecting down to the k spaces of embeddings of $(k-1)$ points in $\mathbb{R}^d$ by omitting each of the k points in turn, and then projecting down from each of those spaces to embeddings of $(k-2)$ points and so on. The Example 3.3.7 shows the cube generated by $\operatorname{Emb}\big(\{x_1, x_2, x_3\}, \mathbb{R}^d\big)$.

Proposition 3.3.8 ([**9**]). *The total fiber of the cube generated by*

$$\operatorname{Emb}\big(\{x_1, \dots, x_k\}, \mathbb{R}^d\big)$$

is $(k-1)(d-2)$*-connected.*

Therefore, in general

$$\text{connectivity of } L_k \operatorname{Emb}(M, \mathbb{R}^d) = (k-1)(d-2) - k \cdot \dim(M) \tag{3.3.9}$$

$$= k(d-m-2) + 2 - d. \tag{3.3.10}$$

where $m = \dim(M)$.

We have a fibration sequence

$$L_k \operatorname{Emb}(M, \mathbb{R}^d) \to T_k \operatorname{Emb}(M, \mathbb{R}^d) \to T_{k-1} \operatorname{Emb}(M, \mathbb{R}^d),$$

and the connectivity of the fiber $L_k \operatorname{Emb}(M, \mathbb{R}^d)$ is one less than the connectivity of the fibration $T_k \operatorname{Emb}(M, \mathbb{R}^d) \to T_{k-1} \operatorname{Emb}(M, \mathbb{R}^d)$. From Equation 3.3.10 above, we know that the connectivity of $L_k \operatorname{Emb}(M, \mathbb{R}^d)$ is $k(d-m-2)+2-d$, and therefore the map $T_k \operatorname{Emb}(M, \mathbb{R}^d) \to T_{k-1} \operatorname{Emb}(M, \mathbb{R}^d)$ is $(k(d-m-2)+3-d)$-connected.

In the case of embeddings, *disjunction results* from [**6**] in addition to the fact that L_k is $(k(d-m-2)+2-d)$-connected tell us that the map $\operatorname{Emb}(M, \mathbb{R}^d) \to T_k \operatorname{Emb}(M, \mathbb{R}^d)$ is $(k(d-m-2)+1-m)$-connected. Therefore, as $k \to \infty$, the connectivity of $\operatorname{Emb}(M, \mathbb{R}^d) \to T_k \operatorname{Emb}(M, \mathbb{R}^d)$ also goes to ∞. We conclude that the Taylor tower for embeddings converges to $\operatorname{Emb}(M, \mathbb{R}^d)$.

CHAPTER 4

Applications of the Manifold Calculus to the Topological Tverberg Conjecture

4.1. Motivation

The Taylor tower for $\mathrm{Emb}(M, \mathbb{R}^d)$ approximates the embeddings functor through successively better and better approximations. The first stage of the tower, immersions, is the "worst" approximation (see Theorem 3.3.1), because it allows maps to have infinitely many self-intersections. The second stage of the tower can be thought of as maps from which we can remove two-fold intersections, but not three-fold or higher. Similarly, at the kth stage of the tower, we have maps from which we can remove k-fold intersections, but not n-fold intersections for $n > k$. For more precise statements, see [**8**].

Recall Conjecture 2.2.1: For integers $r \geq 2$ and $d \geq 1$, there does not exist an almost r-immersion $f : \Delta^{(d+1)(r-1)} \longrightarrow \mathbb{R}^d$. The fact that the Taylor tower for embeddings is trying to remove self-intersections from maps suggests that the manifold calculus could be related to the Topological Tverberg Conjecture, since in the conjecture we are interested in removing r-fold self-intersections from maps.

Our eventual goal is to use the tools involved in showing the convergence of the Taylor tower for the embeddings functor to find and classify counterexamples to the Topological Tverberg Conjecture. In the case of embeddings, we forbid maps from having 2-fold intersections, while in the Tverberg problem, we forbid maps $f : \Delta^N \to \mathbb{R}^d$ from having r-fold intersections unless the intersection results from faces in Δ^N that are not pairwise disjoint. Definition 2.1.5 gives us the functor we plan to work with: the space of almost r-immersions, which we will denote $r\,\mathrm{Imm}_0(M, \mathbb{R}^d)$.

It is easier to work with r immersions than with almost r-immersions, since for r-immersions there are no additional conditions on where r-fold intersections come from in the initial space. We claim that the Taylor tower for r-immersions is related to that of embeddings. The space of 2-immersions is precisely the space of embeddings, so the two spaces have the same Taylor tower. The Taylor tower for r-immersions should also be related to that of $\mathrm{Emb}(M, \mathbb{R}^d)$.

During this thesis we do not have time to consider the case of almost r-immersions, but we are able to formulate the strategy of a proof for the Taylor tower of r-immersions, which should be similar. We can also say something about low stages of the Taylor tower of $r\,\mathrm{Imm}(M, \mathbb{R}^d)$.

4.2. Results

PROPOSITION 4.2.1. *Let M be a smooth manifold. Then*

$$T_1 r\,\mathrm{Imm}(M,\mathbb{R}^d) \simeq \mathrm{Imm}(M,\mathbb{R}^d).$$

Furthermore, the map

$$r\,\mathrm{Imm}(M,\mathbb{R}^d) \longrightarrow T_1 r\,\mathrm{Imm}(M,\mathbb{R}^d).$$

is at least $(d-2m-1)$-connected.

PROOF. The fact that the first stage of the Taylor tower for $r\,\mathrm{Imm}(M,\mathbb{R}^d)$ is homotopy equivalent to $\mathrm{Imm}(M,\mathbb{R}^d)$ comes from the fact that the first stage of the Taylor tower for $\mathrm{Emb}(M,\mathbb{R}^d)$ is homotopy equivalent to $\mathrm{Imm}(M,\mathbb{R}^d)$. The argument that

$$T_1\,\mathrm{Emb}(M,\mathbb{R}^d) \simeq \mathrm{Imm}(M,\mathbb{R}^d)$$

(which can be found in [**10**], Example 10.2.17) holds for r-immersions as well. Roughly, the key is that the spaces of r-immersions and immersions are homotopy equivalent on single disks.

To see the connectivity estimate, consider the following commutative square:

$$\begin{array}{ccc} \mathrm{Emb}(M,\mathbb{R}^d) & \longrightarrow & T_1\,\mathrm{Emb}(M,\mathbb{R}^d) \\ \downarrow & & \downarrow \\ r\,\mathrm{Imm}(M,\mathbb{R}^d) & \longrightarrow & T_1 r\,\mathrm{Imm}(M,\mathbb{R}^d). \end{array}$$

We know that the top horizontal map is $(d-2m-1)$-connected (from Theorem 3.3.2 with $k=1$) and that the right vertical map is an equivalence. Therefore, their composition is at least $(d-2m-1)$-connected. The composition of the left vertical map and the bottom horizontal map is also at least $(d-2m-1)$-connected, which means that each of the left vertical map and the bottom horizontal map is at least $(d-2m-1)$-connected. Thus the map

$$T_1 r\,\mathrm{Imm}(M,\mathbb{R}^d) \longrightarrow \mathrm{Imm}(M,\mathbb{R}^d).$$

is at least $(d-2m-1)$-connected. □

We conjecture that the connectivity of the map

$$T_1 r\,\mathrm{Imm}(M,\mathbb{R}^d) \longrightarrow \mathrm{Imm}(M,\mathbb{R}^d)$$

depends on r as well as on d and m. See the discussion following Conjecture 4.2.3 for more details. The following theorem describes the next $r-1$ stages of the Taylor tower.

THEOREM 4.2.2. *For integers $2 \le k \le (r-1)$, we have*

$$T_k r\,\mathrm{Imm}(M,\mathbb{R}^d) \simeq T_1 r\,\mathrm{Imm}(M,\mathbb{R}^d).$$

PROOF. Let $2 \le k \le (r-1)$, and consider $L_k r\,\mathrm{Imm}(M,\mathbb{R}^d)$. As with embeddings, the connectivity of this space is determined by the total fiber of a k cube that starts with $r\,\mathrm{Imm}\left(\{x_1,\ldots,x_k\},\mathbb{R}^d\right)$, and projects down to $k-1$ points in k directions and so on (see the cube in Example 3.3.7). Note that these projection maps are

$$r\,\mathrm{Conf}\left(m,\mathbb{R}^d\right) \longrightarrow r\,\mathrm{Conf}\left(m-1,\mathbb{R}^d\right)$$

for $0 \leq m \leq k$ (where $r\,\mathrm{Conf}(0, \mathbb{R}^d) \simeq *$). Since $m \leq k \leq (r-1)$, these spaces are contractible by Example 1.1.9. Recall from Example 1.2.8 that the homotopy fiber of a map between two contractible spaces is also contractible. From Definition 1.2.12, we can compute the connectivity of the total homotopy fiber of the cube generated by $r\,\mathrm{Imm}\big(\{x_1, \ldots, x_k\}, \mathbb{R}^d\big)$ iteratively by viewing the cube as a map of lower-dimensional cubes. Since all of the spaces in the original k-cube are contractible, taking the homotopy fibers of the projection maps in the cube first in one direction yields a $(k-1)$-cube of contractible spaces. Repeating this process iteratively, the total homotopy fiber of the k-cube generated by $r\,\mathrm{Imm}\big(\{x_1, \ldots, x_k\}, \mathbb{R}^d\big)$ is contractible, and therefore infinitely connected.

From Proposition 3.3.5, we know that the connectivity of $L_k F$ is the connectivity of the total homotopy fiber of the cube generated by $r\,\mathrm{Imm}\big(\{x_1, \ldots, x_k\}, \mathbb{R}^d\big)$ minus km. As shown above, the total homotopy fiber of the cube in question is infinitely connected, so in this case L_k is infinitely connected. Thus L_k is (weakly) contractible.

Since

$$L_k r\,\mathrm{Imm}(M, \mathbb{R}^d) = \mathrm{hofiber}(T_k r\,\mathrm{Imm}(M, \mathbb{R}^d) \longrightarrow T_{k-1} r\,\mathrm{Imm}(M, \mathbb{R}^d))$$

is contractible, we know that the homotopy fiber

$$\mathrm{hofiber}(T_k r\,\mathrm{Imm}(M, \mathbb{R}^d) \longrightarrow T_{k-1} r\,\mathrm{Imm}(M, \mathbb{R}^d))$$

is also contractible. Then $T_k r\,\mathrm{Imm}(M, \mathbb{R}^d) \simeq T_{k-1} r\,\mathrm{Imm}(M, \mathbb{R}^d)$ for $2 \leq k \leq r-1$, so $T_k r\,\mathrm{Imm}(M, \mathbb{R}^d) \simeq T_1 r\,\mathrm{Imm}(M, \mathbb{R}^d)$ for all $2 \leq k \leq r-1$. □

Note that Theorem 4.2.2 does not show that the map

$$r\,\mathrm{Imm}(M, \mathbb{R}^d) \longrightarrow T_k r\,\mathrm{Imm}(M, \mathbb{R}^d)$$

is an equivalence for $2 \leq k \leq (r-1)$.

In general, in order to know the connectivity of the maps from $r\,\mathrm{Imm}(M, \mathbb{R}^d)$ to the stages of the Taylor tower, we would need to know the connectivity of

$$T_k r\,\mathrm{Imm}(M, \mathbb{R}^d) \longrightarrow T_{k-1} r\,\mathrm{Imm}(M, \mathbb{R}^d),$$

as well as *disjunction results* similar to those in [**6**] which were used for the connectivity in the case of embeddings. This appears to be a hard problem.

CONJECTURE 4.2.3. *For any* $k \in \mathbb{Z}_{\geq 1}$, *let* $s = \lfloor \frac{k}{r-1} \rfloor + 1$. *Then*

$$T_k r\,\mathrm{Imm}(M, \mathbb{R}^d) \simeq T_s\,\mathrm{Emb}(M, \mathbb{R}^d).$$

Theorem 4.2.2 is a special case of Conjecture 4.2.3. Recall that

$$2\,\mathrm{Imm}(M, \mathbb{R}^d) = \mathrm{Emb}(M, \mathbb{R}^d),$$

and therefore the two have the same Taylor tower. From Theorem 4.2.2, the first $r-1$ stages of the Taylor tower for $r\,\mathrm{Imm}(M, \mathbb{R}^d)$ are the same as the first stage of the Taylor tower for $\mathrm{Emb}(M, \mathbb{R}^d)$. Conjecture 4.2.3 says that the higher stages of the Taylor tower for $r\,\mathrm{Imm}(M, \mathbb{R}^d)$ are the stages for $\mathrm{Emb}(M, \mathbb{R}^d)$ "spread out" by multiples of $r-1$.

Furthermore, from Proposition 4.2.1 the connectivity of the map

$$r\,\mathrm{Imm}(M, \mathbb{R}^d) \longrightarrow \mathrm{Imm}(M, \mathbb{R}^d)$$

is related to that of

$$\mathrm{Emb}(M, \mathbb{R}^d) \longrightarrow \mathrm{Imm}(M, \mathbb{R}^d).$$

Because we believe that the Taylor tower for $r\,\mathrm{Imm}(M,\mathbb{R}^d)$ is the Taylor tower for $\mathrm{Emb}(M,\mathbb{R}^d)$ scaled by $r-1$, we also believe that the connectivities of the maps

$$r\,\mathrm{Imm}(M,\mathbb{R}^d) \longrightarrow T_k r\,\mathrm{Imm}(M,\mathbb{R}^d)$$

are related to the connectivities of the maps

$$\mathrm{Emb}(M,\mathbb{R}^d) \longrightarrow T_k\,\mathrm{Emb}(M,\mathbb{R}^d)$$

by some relation that depends on r. In other words, we conjecture that the analog of Theorem 3.3.2 for $r\,\mathrm{Imm}(M,\mathbb{R}^d)$ has connectivities that depend on r in addition to m and d. When $r=2$, the connectivities become those in Theorem 3.3.2. Since the Taylor tower for $\mathrm{Emb}(M,\mathbb{R}^d)$ converges and the Taylor tower for r-immersions has the same structure, the Taylor tower for $r\,\mathrm{Imm}(M,\mathbb{R}^d)$ should also converge.

4.3. Back to the Topological Tverberg Conjecture

It is known that the Topological Tverberg Conjecture is true for r a power of a prime, and false in other cases. For example, the conjecture is false when r is not a power of a prime and $d \geq 2r-1$. Therefore, results for a manifold calculus of functors solution to the Topological Tverberg Conjecture should depend on whether r is a power of a prime. However, the above conjectures for the Taylor tower for $r\,\mathrm{Imm}(M,\mathbb{R}^d)$ do not depend on whether r is a power of a prime. In the proof that the conjecture holds for r a power of a prime, the requirement that r be a power of a prime comes from the "almost" part of the definition of almost r-immersions (see, for example, [**7**]), which is the piece that our work has yet to take into account. Therefore, the additional step of considering the space of almost r-immersions should yield information about when the Topological Tverberg Conjecture holds and when it does not.

We will denote the space of almost r-immersions by $r\,\mathrm{Imm}_0(M,\mathbb{R}^d)$.

REMARK 4.3.1. We can consider the space of almost r-immersions of a manifold M by first considering a triangulation of M and then applying Definition 2.1.5 to this triangulation.

For information about the conjecture, convergence of the Taylor tower for almost r-immersions is not necessary. Rather, future work should focus on establishing some sort of connectivity

$$\alpha : r\,\mathrm{Imm}_0(M,\mathbb{R}^d) \longrightarrow T_k r\,\mathrm{Imm}_0(M,\mathbb{R}^d).$$

If, at any stage of the tower, this map has connectivity greater than or equal to 0, this would tell us that α induces a surjection

$$\pi_0(r\,\mathrm{Imm}_0(M,\mathbb{R}^d)) \longrightarrow \pi_0(T_k r\,\mathrm{Imm}_0(M,\mathbb{R}^d)).$$

Then, if $T_k r\,\mathrm{Imm}_0(M,\mathbb{R}^d) \neq \emptyset$, we would have $r\,\mathrm{Imm}_0(M,\mathbb{R}^d) \neq \emptyset$ as well. If $r\,\mathrm{Imm}_0(M,\mathbb{R}^d) \neq \emptyset$, then there exists an almost r-immersion of M in $\mathbb{R}^d$, yielding a counterexample to the Topological Tverberg Conjecture.

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