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Generalizations of Nil Clean to Ideals

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Abstract

The notion of a clean ring has many variations that have been widely studied, including the sub-class of nil clean rings. We develop new variations of this concept and discuss the interactions between these new properties and those in the established canon. The first property we define is an ideal-theoretic generalization of the element-wise defined property "nil clean," the condition that an element of a ring is the sum of a nilpotent and an idempotent. We establish a few characterizations for certain families of rings with this property, called ideally nil clean. In particular, a commutative ring is ideally nil clean if and only if it is strongly π-regular. We show that the class of ideally nil clean rings also includes artinian rings, and von Neumann regular rings. Among other results, we demonstrate that ideally nil clean rings behave well under some ring extensions such as direct products and matrix rings. We also expand this generalization to the ideally nil clean property for one-sided ideals, and discuss the interaction between these different generalizations. We explore the interplay between nil clean rings and ideally nil clean rings.

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Contents

CHAPTER 1

Introduction

Over the past century and a half, the theory of rings has played a significant role in the development of mathematics for many reasons, including as a means to explore the interaction of the operations addition and multiplication under either highly specific or very general conditions. A prominent area of ring theory is the study of elements with special multiplicative properties. A nilpotent element n of ring R has the property that $n^k = 0$ for some positive integer k. An idempotent element e of ring R satisfies $e^2 = e$. Another significant facet of the theory of rings concerns properties of ideals, which can be very different from those of elements, or remarkably similar. We say an ideal $\mathfrak{A}$ in ring R is nilpotent if $\mathfrak{A}^k = (0)$ for some positive integer k, and ideal $\mathfrak{E}$ is idempotent if $\mathfrak{E}^2 = \mathfrak{E}$. An ideal $\mathfrak{N}$ is called nil if every element of $\mathfrak{N}$ is nil. We will observe that elements and ideals with these multiplicative properties interact quite differently. Throughout this paper rings are associative with unity.

In 1977, W. K. Nicholson defined a ring R to be clean if for every $a \in R$ there is u a unit in R and e an idempotent in R such that $a = u + e$ [7]. The interest in the clean property of rings stems from its close connection to exchange rings, since clean is a nice property that implies exchange, and there are few exchange rings that are not clean. Clean rings behave well in a lot of ways; for instance, a matrix ring over a clean ring is also clean. Yet in other ways, clean rings do not; for example, the corner ring of a clean ring need not be clean [8]. Properties of rings related to the clean and exchange properties have been largely expanded and researched, and some closely relate to other properties of interest to algebraists. For instance, there are close connections between strongly clean rings, rings in which every element a is the sum of an idempotent and unit that commute with each other, i.e. $a = e + u$ and $eu = ue$, and certain rings that are defined in terms of multiplicative properties of elements.

John von Neumann defined a von Neumann regular ring R as a ring in which for every element $x \in R$ there is an element $y \in R$ such that $x = xyx$ [3]. In an arbitrary ring, any element with this property is called von Neumann regular as well. Von Neumann first established this definition in order to prove results about operator algebras and various structure relevant to the study of functional analysis. However, since then von Neumann regular rings have been studied as an interesting class of rings in their own right. A property that makes von Neumann regular rings interesting and useful in the study of ideals with multiplicative conditions is the fact that every element $x \in R$ has a multiple xy that is an idempotent, which provides information about the ideal that x generates [3]. There are some connections between variations of von Neumann regular rings and variations of clean rings.

In 2013, A. J. Diesl defined a ring R to be nil clean if for every $a \in R$ there is n a nilpotent element in R and e an idempotent in R such that $a = n + e$ [2]. This property is strictly stronger than clean for rings, since he proves every nil clean ring is clean, and provides examples. His work was motivated by connections that the nil clean property has to variations of von Neumann regularity. The interplay of these additive and multiplicative conditions allow for diverse characterizations of various classes of rings. The class of nil clean rings includes all boolean rings and matrices over any nil clean ring, but is limited in scope in some ways, since for instance every nil clean ring has characteristic two [2].

We generalize the nil clean property to an additive decomposition of ideals rather than elements by defining an ideal $\mathfrak{A}$ in a ring R to be "ideally nil clean" if it can be written as the sum of an idempotent ideal and a nil ideal. We provide a set of examples of ideally nil clean rings that serve to demonstrate the intersections or distinctions between ideally nil clean rings and variations of von Neumann regular rings, artinian rings, clean rings, and nil clean rings, culminating in an unresolved conjecture concerning the latter. We demonstrate that ideally nil clean rings behave well under some ring extensions, and include many familiar classes of rings including fields and finite rings. In the process of generating this partial characterization, we establish an interesting result about nil clean rings as well. We discuss the obstructions that currently prevent a complete characterization of INC rings.

CHAPTER **2**

Background

2.1 Background on Chain Conditions and Radicals

One of the defining properties that characterized early research in ring theory is the ascending chain condition. We say that a chain of modules over a ring $M_1 \subseteq M_2 \subseteq \cdots$ satisfies the ascending chain condition if there exists a positive integer k such that $M_i = M_{i+1}$ for all $i > k$. Rings in which every chain of left ideals satisfies the ascending chain condition are called left noetherian, and right noetherian rings are defined analogously.

Similarly, we say a chain of modules over a ring $M_1 \supseteq M_2 \supseteq \cdots$ satisfies the descending chain condition if there exists a positive integer k such that $M_i = M_{i+1}$ for all $i > k$. Rings in which every chain of left ideals satisfies the descending chain condition are called left artinian, and right artinian rings are defined analogously. If a ring satisfies both conditions, it is simply called artinian. The Hopkins-Levitzski theorem implies, among other things, that all artinian rings are noetherian. Not all noetherian rings are aritnian however. In fact, the integers are easily seen to be a noetherian ring that is not artinian. The class of artinian rings includes all finite rings, as well as the ring of $n \times n$ matrices over any artinian ring R, denoted $\mathbb{M}_n(R)$ [6].

A structure of ring theory related to ideals are modules, which are closed under addition and stable under multiplication by elements of the ring, but need not be a subset of the ring. Rather than considering the ideals of a module, we consider its submodules. A module over a ring is said to be simple if its only submodules are trivial. Similarly, a ring R is said to be simple if its only (two-sided) ideals are (0) and R. Recall that if a commutative ring is simple, then it is a field. A module is semisimple if it is the direct sum of simple modules. Similarly R is called semisimple if it is semisimple as a module over itself, or equivalently, it is semisimple if it is the finite direct sum of simple artinian rings. Whereas not all simple rings are artinian, though commutative simple rings certainly are, there is a useful and complete characterization of simple artinian rings. In fact, the characterization of simple artinian rings is a direct corollary of a result characterizing artinian semisimple rings known as the Wedderburn-Artin theorem. The Wedderburn-Artin theorem states that every artinian semisimple ring is isomorphic to $\mathbb{M}_{n_1}(D_1) \times \cdots \times \mathbb{M}_{n_k}(D_k)$

for some positive integers $n_1, \ldots, n_k$ and some division rings $D_1, \ldots, D_k$ [6].

To further understand the structure of rings, we will briefly discuss the Jacobson radical and nil radical of a ring. Recall that in a commutative ring, the set of nilpotents forms an ideal of the ring, denoted $Nil(R)$. In a noncommutative ring, the set of nilpotents need not form an ideal, yet there are many ways to meaningfully generalize the nil radical. For instance, we define $Nil^*(R)$ to be the sum of all nil ideals, which is itself a nil ideal. However, it may in fact be a much smaller set than the set of all nilpotents. Therefore there are other radicals that are used to capture similar properties for noncommutative rings to those for which the nil radical is used in the commutative case. The Jacobson radical of a ring is defined to be the intersection of all maximal right ideals, which is also equal to the intersection of all maximal left ideals, and is itself an ideal of the ring. It can also be characterized as the set of all elements $x \in R$ such that for all $y, z \in R$ the element $1 - zxy$ is a unit. We denote the Jacobson radical of a ring R by $J(R)$ [6].

Observe that many properties of elements in $R/J(R)$ are true of their preimages in R, and when this holds we say that elements with that property "lift modulo the radical." For example, if $\bar{u}$ is a unit in $R/J(R)$, then $\bar{u}\bar{u}^{-1} = \bar{1}$ which means that if u is in the preimage of $\bar{u}$, there exists $b \in J(R)$ such that $uu^{-1} = 1 + b$. Then because $1 + b$ must be a unit, we conclude $uu^{-1}(1 + b)^{-1} = 1$, namely that u is a unit in R. If N is a nil ideal, then a nilpotent element $\bar{n}$ in R/N lifts to nilpotent elements in R. Similarly, for every idempotent element $\bar{e}$ in R/N there exists an idempotent element $e \in R$ to which it lifts [6]. It is a crucial fact that every central idempotent also lifts to a central idempotent modulo a nil ideal, but in general if two elements of R/N or $R/J(R)$ commute, it is not necessary that their preimages under the quotient commute.

2.2 Background on Von Neumann Regularity

One family of properties that will prove to have remarkable connections to the nil clean and ideally nil clean conditions is the collection of variations of von Neumann regular rings. There are a few variations of particular interest from the perspective of nil clean rings, and these properties have even closer ties to ideally nil clean rings. We will state some of the relationships these various properties have to each other, as well as giving some useful alternative characterizations of each property.

The following characterizations of von Neumann regular rings (hereafter called regular rings for brevity) will be very useful to us.

Theorem 2.2.1. *[3, Theorem 1.1] For a ring R, the following are equivalent:*
(1) R is regular.
(2) Every principal right (left) ideal of R is generated by an idempotent.
(3) Every finitely generated right (left) ideal of R is generated by an idempotent.

We now introduce a few variations of regular rings that are particularly relevant to our subsequent line of inquiry.

Definition 2.2.2. *Let R be a ring with unity.*
(1) An element $x \in R$ is called unit regular if there exists an element $u \in U(R)$ such that $x = xux$.
(2) An element $x \in R$ is called strongly regular if there exists an element $y \in R$ such that $x = x^2y = yx^2$.
(3) An element $x \in R$ is called strongly π-regular if there exist elements $r, s \in R$ such that $x^n = rx^{n+1} = x^{n+1}s$ for some natural number $n > 0$.

Equivalently, the element $x \in R$ is called strongly π-regular if $x^n \in Rx^{n+1} \cap x^{n+1}R$ [2]. In particular, strongly regular implies strongly π-regular.

A ring is called unit regular (respectively strongly regular or strongly π-regular) if every element of the ring is unit regular (respectively strongly regular or strongly π-regular.) In [1] it is shown that every unit regular rings is clean, demonstrating the connection between clean and variations of regularity. In [2] the author demonstrates that an element $a \in R$ is strongly π-regular if and only if there exist an idempotent element $e \in R$ and a unit $u \in R$ such that $a = e + u$, $ae = ea$, and eae is a nilpotent. This alternative characterization will be crucial for understanding the connection between strongly π-regular rings and ideally nil clean rings. Observe that as a result of this characterization, strongly π-regular implies strongly clean.

2.3 Preliminary Remarks on Nil and Nilpotent Ideals

Similar to the existence of multiple generalizations of the nil radical to noncommutative rings, there are multiple ways one can generalize the nilpotent property of elements to ideals. For instance there is the direct generalization, that the ideal $\mathfrak{N}$ has the property that $\mathfrak{N}^k = (0)$ for some positive integer k, in which case we call $\mathfrak{N}$ nilpotent. However, one can also generalize by considering ideals in which every element is nilpotent, which is a strictly weaker property. The latter type of ideal is called nil, rather than nilpotent. We will include a thorough background reviewing key results concerning nil and nilpotent ideals. The following results are established in the ring theory canon, and a more rich and vast treatment of nilpotents and radicals can be found in [6]. The following known results will assist us in our study of ideally nil clean rings and its variations.

Definition 2.3.1. *An ideal $\mathfrak{A}$ of a ring R is called nilpotent if there exists a positive integer k such that $\mathfrak{A}^k = (0)$.*

The following basic results can be found in [6] or in more elementary sources on ring theory.

Proposition 2.3.2. *Every nilpotent ideal is a nil ideal.*

Proof. Suppose $\mathfrak{A}$ is a nilpotent ideal with index of nilpotence $k > 0$. Then we know the product of any k elements of $\mathfrak{A}$ is zero, so for all $a \in \mathfrak{A}$ we know $a^k = 0$. $\square$

Although the arbitrary sum of nilpotent ideals need not be nilpotent, it is true that the arbitrary sum of nil ideals is nil. This is because if $a \in \sum_{i\in\alpha} N_i$ where α is an arbitrary ordinal, then by definition of ideal summations we know that a is the finite sum of elements from only a finite subcollection $\{N_k\} \subseteq \{N_i\}_{i\in\alpha}$ of nil ideals in the collection. Thus a is contained in a finite sum of nil ideals, which we know to be a nil ideal.

Observe that not all rings with a bounded index of nilpotence have the property that every nil ideal is nilpotent, but every ring in which all nil ideals are nilpotent must have a bound for the index of nilpotence of its elements. Otherwise, suppose there were no bound for the index of the collection of nilpotents $N = \{n_1, n_2, \ldots\}$ in commutative ring R. Then observe that the ideal generated by the collection N is nil, yet not nilpotent since for all $k > 0$ there exists $n_{k+1} \in N$ such that $n_{k+1}^k \neq 0$.

Example 2.3.3. *Consider $R = \mathbb{Z}_2[x_i]_{i=1}^{\infty}$ and, consider also the ideal in R generated by $\{x_i^i\}_{i=1}^{\infty}$, denoted $(\{x_i^i\}_{i=1}^{\infty})$. The quotient ring $\mathbb{Z}_2[x_i]_{i=1}^{\infty}/(\{x_i^i\}_{i=1}^{\infty})$ has infinitely many nilpotents with arbitrarily high index of nilpotence, thus $J(R)$ is a nil ideal that is not nilpotent, since $J(R)$ contains all polynomials with constant term zero, but no polynomials with constant term one.*

In a commutative ring, all collections of nilpotent elements generate nil ideals, and finite collections of nilpotents generate nilpotent ideals. In a noncommutative ring, a nilpotent need not generate a nilpotent or nil ideal (consider, for example, the nilpotent $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ in the ring $\mathbb{M}_2(\mathbb{R})$, which has no nontrival ideals, so the ideal generated by this nilpotent is the ring $\mathbb{M}_2(\mathbb{R})$). We also note that every nil ideal of a ring is contained in the Jacobson radical of the ring R, because $1 + n \in U(R)$ for every nilpotent $n \in R$ [6]. We include the proof of this fact below.

Proposition 2.3.4. *For any ring R, the element $1 + n$ is a unit for every nilpotent $n \in R$*

Proof. If $n^k = 0$, then observe that $1 = 1 - n^k = (1-n)(1+n+n^2+\cdots+n^{k-1})$ and therefore $(1 + n + n^2 + \cdots + n^{k-1})$ is the inverse of $(1 - n)$. If we replace n with $-n$, which has the same index of nilpotence, the result still holds. Therefore $1 + n$ is invertible for any nilpotent $n \in R$. $\square$

This gives the following useful corollary.

Proposition 2.3.5. *Let R be a ring in which u is a unit of R and n is a nilpotent with index of nilpotence $k > 0$. If $un = nu$, then $u + n \in U(R)$.*

Proof. Observe that by the binomial identity, we know $(u + n)^k = u^k + \binom{k}{1}u^{k-1}n + \cdots + \binom{k}{k-1}un^{k-1} + n^k = u^k + \binom{k}{1}u^{k-1}n + \cdots + \binom{k}{k-1}un^{k-1}$. Therefore $u^{-k}(u+n)^k = 1 + u^{1-k}n + \cdots + u^{-1}n^{k-1}$. Since u and n commute we know $u^{1-k}n + \cdots + u^{-1}n^{k-1}$ is a nilpotent. Thus

$1 + u^{1-k}n + \cdots + u^{-1}n^{k-1}$ is a unit, and so is every factor of $u^{-k}(u+n)^k$, namely $u+n$.

□

Similarly, we may consider nil and nilpotent one-sided ideals, which have completely analogous definitions. However, there are significant differences between results established for two-sided ideals and one-sided ideals, and in general the relationship between the existence of nil ideals and one-sided nil ideals is not understood. The obstacles to understanding nil one-sided ideals is illustrated by the following unsolved conjecture.

Köthe's Conjecture. *[6, 10.28] If $Nil^*R = 0$, then R has no nonzero nil one-sided ideals.*

The conjecture in fact has many equivalent formulations, for instance the claim that every nil left ideal or right ideal of a ring R is contained in $Nil^*(R)$, as well as the proposition that the sum of two nil left ideals (or two right ideals) of a ring is also a nil left (right) ideal. Of particular relevance to results we establish in later sections, the conjecture that if I is a nil ideal of a ring R then $\mathbb{M}_n(I)$ is a nil ideal of $\mathbb{M}_n(R)$ for any $n \in \mathbb{Z}_{\geq 1}$ is also equivalent to Köthe's [6]. The conjecture does hold for many important classes of rings including artinian and noetherian rings, so it is meaningful to consider the class of rings for which the Köthe conjecture holds.

2.4 Preliminary Remarks on Idempotent ideals

We must lay some groundwork for our results on nil clean ideals by first making some basic observations about the idempotent property of (two-sided) ideals. The theory of idempotents, while well-established in ring theory canon for elements of rings, is less comprehensively developed as a property defined for ideals.

Definition 2.4.1. *An ideal I of a ring R is called idempotent if $I^2 = I$.*

Note that as a basic property of idempotent ideals, (0) and R are the equivalent of the "trivial" idempotents. If there is any nontrivial central idempotent e in R then R is isomorphic to the direct product $eR \times (1-e)R$ which are both ideals in R and subrings with identities e and $(1-e)$ respectively. Yet a ring can have idempotent ideals other than direct factors. We include the following foundational results concerning idempotent ideals for completeness, as we make frequent use of them.

Proposition 2.4.2. *The sum of idempotent ideals is idempotent.*

Proof. Let $\{E_i\}_{i\in\alpha}$ be an arbitrary set of idempotent ideals in a ring R. Then observe that for all $x \in \sum_{i\in\alpha} E_i$ there is a finite collection of elements $\{e_i | e_i \in E_j, j \in \alpha\}_{i=1}^n$ such that $x = \sum_{i=1}^n r_i e_i s_i = \sum_{i=1}^n r_i f_i g_i s_i$ where $f_i, g_i \in E_j$ if $e_i \in E_j$. Thus we see $x \in \sum_{i\in\alpha} E_i^2$. So $\sum_{i\in\alpha} E_i = \sum_{i\in\alpha} E_i^2$, therefore the ideal is idempotent.

□

Proposition 2.4.3. *If ideal E in R is generated by idempotent elements, then E is an idempotent ideal*

Proof. Since it is always true that $E^2 \subseteq E$, we need only demonstrate that $E \subseteq E^2$. If E is an ideal generated by idempotent elements, then for each $a \in E$ there is a subcollection of the idempotent generators of E, call it $\{e_i\}_{i\in\alpha}$, such that

$$a = \sum_{i\in\alpha} r_i e_i s_i \text{ with } r_i, s_i \in R.$$

Thus, we can rewrite a as

$$a = \sum_{i\in\alpha} r_i e_i e_i s_i = \sum_{i\in\alpha} x_i y_i \text{ where } x_i, y_i \in E$$

so $a \in E^2$. □

We will remark that not all idempotent ideals are generated by idempotent elements. In particular, the following example is a construction of a noncommutative domain in the exercises of [6] that has a single nontrivial ideal, and since the ring is a domain the ideal cannot be generated by an idempotent element, since the only idempotent elements of a domain are 0 and 1.

Example 2.4.4. *[5, Ex. 12.2] Let k_0 be a field of characteristic zero, and let R be the (first) Weyl algebra over k_0, with generators x, y and the relation $xy - yx = 1$, which we may also denote $k_0 < x, y > /(xy - yx - 1)$. Consider the ideal $Rx = \{a_1 x + \cdots + a_n x^n | a_i \in k_0 < x, y >\}$ which one can show is in fact a simple domain without identity (see [5] for the details of this proof). Then the ring $A = Rx \oplus k_0 = \{a_0 + a_1 x + \cdots + a_n x^n | a_0 \in k_0, a_i \in k_0 < x, y >, i \geq 1\}$. Since we know Rx is a nonunital simple domain, we conclude that Rx is the sole nontrivial ideal of A since any other nonzero ideal must contain an element of k_0. Therefore, since A has precisely three ideals and no nontrivial nilpotent elements (a domain can have no nonzero nilpotents) it must be that $(Rx)^2 = Rx$. So Rx is an idempotent ideal not generated by idempotent elements.*

For elements of rings, it is not true in general that the sum of two idempotent elements need be idempotent, nor that their product need be idempotent. However, in certain cases the product of idempotents is idempotent, and we will demonstrate a similar result for idempotent ideals.

Proposition 2.4.5. *The product of idempotent ideals I and J is idempotent if R is a commutative ring or if I and J are generated by central idempotents.*

Proof. If R is commutative, then $(IJ)^2 = IJIJ = I^2J^2 = IJ$ because the elements of I and J commute. One can write the elements of $IJIJ$ as the product of two sums of products of elements and show that by moving around elements of I and J within the sum we get the sum of elements in I^2J^2; this is left as an exercise for the reader.

If I and J are generated by central idempotents, then for each $a \in I$ and $b \in J$ there are collections of idempotents $\{e_i\}_{i\in\alpha}$ and $\{f_i\}_{i\in\alpha}$ such that

$$a = \sum_{i\in\alpha} r_i e_i s_i \text{ and } b = \sum_{j\in\alpha} r_j f_j s_j \text{ with } r_i, r_j, s_i, s_j \in R.$$

Therefore

$$ab = \sum_{i,j\in\alpha} r_i e_i e_i s_i r_j f_j f_j s_j = \sum_{i,j\in\alpha} r_i e_i s_i f_j r_j e_i s_j f_j$$

since all elements of $\{e_i\}_{i\in\alpha}$ and $\{f_i\}_{i\in\alpha}$ are central. Now we need only observe for each i, we know $(r_i e_i)(s_i f_j)(r_j e_i)(s_j f_j) \in IJIJ = (IJ)^2$. Thus $IJ \subseteq (IJ)^2$ and therefore IJ is an idempotent ideal. $\square$

For elements of a ring, the notion of a periodic element, an element $a \in R$ such that $a^n = a$ for some $n \in \mathbb{Z}_{\geq 1}$, is more general than the notion of an idempotent element. However, the same definition of periodic for ideals is equivalent to the definition of idempotent ideals, as shown below.

Proposition 2.4.6. *The following are equivalent for an ideal $I \subseteq R$:*
(1) $I^2 = I$
(2) $I = I^k$ for some $k \in \mathbb{Z}_{>1}$.

Proof. $(1) \implies (2)$ If I is idempotent then (2) clearly holds.
$(2) \implies (1)$ If there is a positive integer $k \geq 2$ such that $I^k = I$, then observe $I \subseteq I^k \subseteq I^2$, and it is always true that $I^2 \subseteq I$, therefore $I^2 = I$. $\square$

Therefore, we will generalize the notion of periodicity to ideals in a different way.

Definition 2.4.7. *We say an ideal I in a ring R is periodic if there exists some k such that $I^k = I^{k+1}$.*

Observe that both idempotent ideals and nilpotent ideals fit this definition, as do all ideals in an artinian ring. However, there are non-artinian rings in which every ideal is periodic, as Example 2.4.4 shows, since all artinian domains are simple.

CHAPTER 3

Examples and Ring Extensions of INC Rings

3.1 Examples of INC Rings

We will begin with a formal definition.

Definition 3.1.1. *Let R be a ring. An ideal $A \in R$ is called ideally nil clean (or INC) if there is an idempotent ideal $E \subseteq R$ and a nil ideal $N \subseteq R$ such that $A = E + N$. A ring R is called ideally nil clean (INC) if every ideal in R is ideally nil clean.*

This property is a direct generalization of the nil clean property for elements of a ring defined in [2]. Since there are natural analogs in canonical ring theory for both nilpotent elements and idempotent elements, this variation of the nil clean property provides a natural class of rings to study. If it were the case that nilpotent and idempotent ideals behaved precisely as nilpotent and idempotent elements do, this variation would perhaps be far less compelling. However, for ideals, summation and multiplication have very different effects than those operations for elements of the ring. In fact, we will see that the nil clean property for ideals encompasses a rather more vast collection of rings, and has particularly intriguing interplay with many canonical classes of rings whose intersection with nil clean rings are less prevalent.

To understand the effect of the different properties that addition and multiplication have for ideals, we are motivated to make the following observations.

Lemma 3.1.2. *The sum of INC ideals is INC.*

Proof. This proof is nearly identical to that of the above lemma. Suppose $I_\alpha = N_\alpha + E_\alpha$ is an INC decomposition of the ideal $I_\alpha \subseteq R$ for all $\alpha \in \gamma$ where γ is any ordinal. Then $\sum_{\alpha\in\gamma} I_\alpha = \sum_{\alpha\in\gamma} N_\alpha + \sum_{\alpha\in\gamma} E_\alpha$. By prior lemmata we know $\sum_{\alpha\in\gamma} N_\alpha$ is nil since N_α is nil for all α, and $\sum_{\alpha\in\gamma} E_\alpha$ since E_α is idempotent for all α. $\square$

Corollary 3.1.3. *If every principal ideal in R is INC, then every ideal in R is INC.*

Proof. If for all $a \in I$ we know $(a) = N + E$ where N is a nil ideal and E is an idempotent ideal, then observe that $I \subseteq \cup_{a \in I}(a)$ and $(a) \subseteq I$ for all $a \in I$, therefore $I = \sum_{a \in I} N_a + \sum_{a \in I} E_a$. Since we have shown that the arbitrary sum of idempotent ideal is idempotent, and it is well known that the arbitrary sum of nil ideals is nil, I is ideally nil clean. $\square$

Prior to establishing any further results, we might naturally wonder whether the class of INC rings is trivial, by which we mean, is it possible that every ideal with this property is either nil or idempotent? Or equivalently, are there any INC rings in which there exists I an ideal of R such that I is neither idempotent nor nil as an ideal?

Observe that $\mathbb{Z}_{12}$ is isomorphic to the direct sum of the ideals generated by its nontrivial idempotent elements 9 and 4, i.e. $\mathbb{Z}_{12} \simeq 4\mathbb{Z}_{12} \oplus (1-4)\mathbb{Z}_{12} = 4\mathbb{Z}_{12} \oplus 9\mathbb{Z}_{12}$. Note that $\mathbb{Z}_{12}/4\mathbb{Z}_{12} \simeq \mathbb{Z}_3$ which has no nontrivial ideals, and $\mathbb{Z}_{12}/9\mathbb{Z}_{12} \simeq \mathbb{Z}_4$, which has a single nontrivial nilpotent ideal. This is the underlying reason that, as we will explicitly show, the ring $\mathbb{Z}_{12}$ is INC.

Consider the nontrivial ideals in $\mathbb{Z}_{12}$: We have $(3) = (9)$ and $(4) = (8)$ and $(2) = (10)$ since they are each others' additive inverses. There is also the nontrivial ideal $(6) = \{0, 6\}$ since 6 is the unique nontrivial nilpotent of $\mathbb{Z}_{12}$. Since $(3) = (9)$ and $(4) = (8)$ and (6) are trivially ideally nil-clean, it only remains to show that $(2) = (10)$ is INC as well. Observe that $(2)(2) = (4)$ is not itself an idempotent ideal, and not nil because 2 is not nilpotent. However, $10 = 6 + 4 \in (6) + (4)$ and $2 = 6 + 8 \in (6) + (4)$ so $(2) = (10)$ is contained in $(6) + (4)$ but it is easy to see this means $(2) = (10) = (6) + (4)$. So the above question has an affirmative answer.

Proposition 3.1.4. *The finite direct product of ideally nil clean rings is ideally nil clean.*

Proof. Let $R_1, \ldots, R_k$ be INC rings, and let $R = R_1 \times \cdots \times R_k$. This ring R that results from taking a finite direct product of ideally nil clean rings can only have ideals of the form $\mathfrak{A}_1 \times \mathfrak{A}_2 \times \cdots \times \mathfrak{A}_k$ where $\mathfrak{A}_i$ is an ideal of R_i. We may observe that since each $\mathfrak{A}_i$ is the sum of an idempotent ideal and a nil ideal in R_i, namely $\mathfrak{A}_i = \mathfrak{I}_i + \mathfrak{N}_i$. We may divide the direct product into the sum of the direct product of the summands of each $\mathfrak{A}_i$ that are nil and the direct product of the summands of each $\mathfrak{A}_i$ that are idempotent. Thus we see $\mathfrak{A}_1 \times \mathfrak{A}_2 \times \cdots \times \mathfrak{A}_k = \mathfrak{I}_1 \times \cdots \times \mathfrak{I}_k + \mathfrak{N}_1 \times \cdots \times \mathfrak{N}_k$. The finite direct product of a nil ideal is nil since we may raise any element in the direct product to the max of the indices of nilpotence of the elements in each direct factor, so their product remains nil. Similarly for the direct sum of idempotent ideals, since the direct factors square to themselves, so does the direct product.

$\square$

Example 3.1.5. *The finite direct product of $\mathbb{Z}_{p_i^{e_i}}$ where p_i is a prime for all i. is an example of an ideally nil clean ring.*

Proof. Since $\mathbb{Z}_{p_i^{e_i}}$ is a principal ideal domain, we can easily observe that the only nontrivial ideals in $\mathbb{Z}_{p_i^{e_i}}$ are generated by p_i^k for some $k < e_i$. It is clear that any such ideal is nil, since $(p_i^k)^{e_i} = 0$. Therefore by the above proposition, the finite direct product of such rings is INC. $\square$

In fact, though, it is not difficult to construct a an infinite direct product of ideally nil clean rings that is ideally nil clean. One need only impose an upper bound on the index of nilpotence for all elements of the factors of the direct products. However, if there is no upper bound on the index of nilpotence, it is easily seen that the direct product of infinitely many nil ideals may contain infinite sequences of nilpotents, of the form $b = (n_1, n_2, n_3, \ldots)$, in which the i^{th} nilpotent in the sequence has an index of nilpotence higher than the previous $i - 1$ nilpotents in the sequence. For such an element of the infinite direct product, there is no $k \in \mathbb{N}$ such that $b^k = 0$. So the infinite direct product of nil ideals need not be nil, and similarly the infinite direct product of INC rings need not be INC, unless the index of nilpotence is bounded.

Example 3.1.6. *If $\{R_i\}_{i\in\alpha}$ is an arbitrary collection of ideally nil clean rings for which there exists $n \in \mathbb{N}$ such that $b^n = 0$ for all nilpotent elements $b \in R_i$ for any $i \in \alpha$, then $\Pi_{i\in\alpha} R_i$ is INC.*

Proof. This follows immediately from the above remarks. □

We now begin our comparison and classification of INC rings and variations of regularity. As we will observe, the INC property is far more general than regularity of rings, and bears a close relationship to some of its generalizations. Due to the intimate connection between regular rings and rings in which all ideals are idempotent, the following results are very natural observations.

Remark 3.1.7. *Any regular ring R is ideally nil clean. In fact, every ideal in a regular ring is an idempotent ideal.*

Proof. For all $a \in R$, since $a = axa$ then if $a \in I$ for some ideal I in R, we know $(a)(xa) \in I^2$ thus $a \in I^2$. Since $I \subseteq I^2$ we conclude I is idempotent. □

Proposition 3.1.8. *If $R/J(R)$ is unit regular and $J(R)$ is nil then R is INC.*

Proof. Suppose $\bar{R} := R/J(R)$ is unit regular. Then if $x \in R$, we know that $\bar{x} = \bar{x}\bar{u}\bar{x}$ for some unit $\bar{u} \in \bar{R}$. Since $(\overline{xu}) = (\bar{x})$ and we know that we can lift $\overline{xu}$ to an idempotent in R, call it e, we know that $xu - e = n$, so xu is nil clean. Since units lift modulo $J(R)$ we know $u \in U(R)$, therefore $(xu) = (x)$ is INC. By an earlier lemma this implies R is INC. □

We conjecture that the above proposition holds for regular rings as well. As we later show, the proof follows easily if one replaces the hypothesis that $J(R)$ is nil with the hypothesis that $J(R)$ is nilpotent. As stated, though, the conjecture remains unresolved.

The following proposition captures the utility of the fact that idempotent and nil ideals hold up well under quotients in general. Whether a property of a ring passes to quotients of the ring exposes some aspect of how well the structures it concerns interact with the ideals of the ring. For INC rings, the quotient preserves much of the desired information.

Proposition 3.1.9. *The quotient ring R/A is ideally nil clean if R is ideally nil clean and $A \subseteq J(R)$.*

Proof. If an ideal I in R is INC then $I = N + E$ and N nil implies $N \subseteq J(R)$. Let $\bar{R} = R/A$. Then $\overline{I} = \overline{E} + \overline{N}$. Moreover, since $R/A = (R/J(R))/(A/J(R))$ [6] we know $\overline{N}$ is nil in $\overline{R}$. Moreover, since $E^2 = E$ we know $\overline{E}^2 = \overline{E^2} = \overline{E}$, which means $\overline{I} = \overline{E} + \overline{N}$ is an INC decomposition. □

Finally, we make another observation that demonstrates how well nil ideals and idempotent ideals interact. The following also introduces a fact about periodic ideals that might shed light on the interaction of periodic ideals and INC ideals in future work.

Remark 3.1.10. *If I is an ideal such that $I^k = I^{k+1}$ for some k and A is a nil ideal such that $I = I^k + A$, then for all $m > 0$, $A^m + I^k = I^m$.*

Proof. By induction: Observe $(A + I^k)^2 = A^2 + I^kA + AI^k + (I^k)^2 = A^2 + I^k = I^2$. Assume $A^m + I^k = I^m$. Then $I^{m+1} = I \cdot I^m = (A+I^k)(A^m+I^k) = A^{m+1}+I^kA^m+AI^k+I^k = A^{m+1} + I^k$. □

3.2 INC Ring Extensions

One way to generate a robust collection of examples of ideally nil clean rings is by looking at ring extensions. Ring extensions include matrices over the ring and upper or lower triangular matrices, polynomial rings over the ring and skew polynomial rings over it, as well as many others. For each kind of extension, we might naturally ask whether we know the extension is INC given the base ring is INC, and vice-versa.

Although the one-sided ideals of matrix rings can be very difficult to work with, the two-sided ideals of matrix rings have a desirable relationship with the base ring. The subset $\mathfrak{A} \subseteq \mathbb{M}_n(R)$ is an ideal if and only if there exists an ideal $I \subseteq R$ such that $\mathbb{M}_n(I) = \mathfrak{A}$ [6]. This will allow us to establish notable results about matrix rings over INC rings.

Lemma 3.2.1. *If $\mathfrak{A}$ is an idempotent ideal of R then $\mathbb{M}_n(\mathfrak{A})$ is an idempotent ideal of $\mathbb{M}_n(R)$.*

Proof. Let $\{a_{11}, \ldots, a_{nn}\}$ be any collection of n^2 elements in $\mathfrak{A}$, and thus $a_{ij} = b_{ij}c_{ij}$ for some $b_{ij}, c_{ij} \in \mathfrak{A}$ for all $i, j \leq n$. Consider

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} c_{11} & 0 & \cdots \\ 0 & 0 & \\ \vdots & & \ddots \end{bmatrix} \begin{bmatrix} b_{11} & 0 & \cdots \\ 0 & 0 & \\ \vdots & & \ddots \end{bmatrix} + \begin{bmatrix} 0 & 0 & \cdots \\ c_{21} & 0 & \\ \vdots & & \ddots \end{bmatrix} \begin{bmatrix} b_{21} & 0 & \cdots \\ 0 & 0 & \\ \vdots & & \ddots \end{bmatrix} + \cdots$$

or, more precisely,

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} = \sum_{i,j=1}^{n} AB$$

where the i, j entry of A is c_{ij} and the j, j entry of B is b_{ij}.
This is clearly the sum of elements of $\mathbb{M}_n(\mathfrak{A})^2$, thus $\mathbb{M}_n(\mathfrak{A}) \subseteq \mathbb{M}_n(\mathfrak{A})^2$, so it is an idempotent ideal. $\square$

Proposition 3.2.2. *If R is an INC ring for which the Köthe conjecture holds, then $\mathbb{M}_n(R)$ is INC.*

Proof. If $\mathfrak{I}$ is an ideal of $\mathbb{M}_n(R)$ then there exists an ideal A in R such that $\mathfrak{I} = \mathbb{M}_n(A)$. Then because $A = N + E$ is an INC decomposition, and the above lemma tells us $\mathbb{M}_n(N + E) = \mathbb{M}_n(N) + \mathbb{M}_n(E)$ is also the sum of a nil ideal (since Köthe holds for R) and an idempotent ideal, we conclude $\mathfrak{I}$ is INC. Therefore so is $\mathbb{M}_n(R)$. $\square$

Remark 3.2.3. *If R is a ring for which $\mathbb{M}_n(R)$ is INC, then R is INC.*

Proof. Since it is known that an ideal I in $\mathbb{M}_n(R)$ is the set of matrices over an ideal $\mathfrak{A}$ of R [6], this remark follows since $I = E + N$ implies there are ideals $\mathfrak{I}$ and $\mathfrak{N}$ that are idempotent and nil in R respectively such that $\mathfrak{A} = \mathfrak{I} + \mathfrak{N}$. $\square$

We now consider upper triangular matrix rings, denoted $\mathbb{T}_n(R)$, where R is an INC ring. Every result we derive is equivalently true for lower triangular matrix rings.

Remark 3.2.4. *The ideal $\mathbb{T}_n(I)$ is idempotent if I is idempotent.*

Proof. Let $\{a_{11}, \ldots, a_{nn}\}$ be any collection of $n(n+1)/2$ elements in I, and thus $a_{ij} = b_{ij}c_{ij}$ for some $b_{ij}, c_{ij} \in \mathfrak{A}$ for all $i, j \leq n$. Consider

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ 0 & \ddots & \vdots \\ \vdots & \ddots & a_{nn} \end{bmatrix} = \sum_{j \geq i \geq 1}^{n} A_{i,j} B_{i,j}$$

where the (i^{th}, i^{th}) entry of $A_{i,j}$ is b_{ij} and the (i^{th}, j^{th}) entry of $B_{i,j}$ is c_{ij}. As above, this is clearly the sum of elements of $\mathbb{T}_n(I)^2$, thus $\mathbb{T}_n(I) \subseteq \mathbb{T}_n(I)^2$, so it is an idempotent ideal. $\square$

Corollary 3.2.5. *If $\mathbb{T}_n(R)$ is INC then R is INC.*

Proof. Let A be the set of strictly upper triangular matrices, and observe that A is an ideal of $\mathbb{T}_n(R)$. Hence, we can compute that $\mathbb{T}_n(R)/A \simeq R^n$, which is INC by Corollary 3.1.9. Since a finite direct product is INC if and only if the direct factors are INC, we conclude that R is INC. $\square$

Remark 3.2.6. *If R is INC, then $\mathbb{T}_n(R)$ is INC.*

Proof. Observe that A is not only an ideal of $\mathbb{T}_n(R)$, but also a nilpotent ideal. Thus, by the fact that if N is a nilpotent ideal of R and R/N is INC then R is INC, since $\mathbb{T}_n(R)/A \simeq R^n$, which is INC since R is, we know $\mathbb{T}_n(R)$ itself is INC. $\square$

These extensions allow us to create many noncommutative examples of INC rings. Another interesting ring extension is the ring of endomorphisms of a countably infinite free R module, which can be represented as the ring of matrices over R with countably infinite columns and rows, in which every column has finitely many nonzero entries. We call this the ring of column-finite matrices over R, which is denoted $\mathbb{CFM}_{\mathbb{N}}(R)$. The following fact and its proof were observed for us by A. J. Diesl, and we have included both the observation and proof for completeness.

Lemma 3.2.7. *The ring $\mathbb{CFM}_{\mathbb{N}}(R)$ is not strongly π-regular for any unital ring R.*

Proof. Consider the element of $\mathbb{CFM}_{\mathbb{N}}(R)$ consisting of ones on the superdiagonal, and zeros in every other entry. Namely, consider

$$S = \begin{bmatrix} 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ 0 & 0 & 0 & \ddots \\ \vdots & & \ddots & \ddots \end{bmatrix}$$

Observe that $\mathbb{CFM}_{\mathbb{N}}(R) = End(\bigoplus_{i=1}^{\infty} R)$ i.e., it is the ring of endomorphisms over a countably generated free R-module. Note that for all $n \in \mathbb{Z}_{\geq 1}$ we can see $S^n((r_1, r_2, \ldots, r_n, 0, 0, \ldots)) = 0$, yet if $r_{n+1} \neq 0$ then $S^n((r_1, r_2, \ldots, r_{n+1}, 0, 0, \ldots)) \neq 0$. Now suppose that there were an element $T \in \mathbb{CFM}_{\mathbb{N}}(R)$ such that $S^n = TS^{n+1}$. This would imply that $TS^{n+1}((r_1, r_2, \ldots, r_{n+1}, 0, 0, \ldots)) = S^n((r_1, r_2, \ldots, r_{n+1}, 0, 0, \ldots)) \neq 0$, however since $S^{n+1}((r_1, r_2, \ldots, r_{n+1}, 0, 0, \ldots)) = 0$ we also know $TS^{n+1}((r_1, r_2, \ldots, r_{n+1}, 0, 0, \ldots)) = 0$, a contradiction. So we conclude $RS^{n+1} \neq RS^n$ for any n. □

Since we will show that for commutative rings INC and strongly π-regular are equivalent conditions, we naturally might ask, is the class of INC rings the same as the class of strongly π-regular rings? Are they always equivalent? As the following remark demonstrates, the answer to this question is a negative.

Remark 3.2.8. *Let K be any field. Then $\mathbb{CFM}_{\mathbb{N}}(K)$ is an INC ring that is not strongly π-regular.*

Proof. It is known that $\mathbb{CFM}_{\mathbb{N}}(K)$ only has one ideal that is not trivial, i.e. not (0) nor the ring itself. This lone nontrivial ideal is the ideal of all matrices that are zero in every row after some finite number, and zero in every column after some other finite number. This ideal must be idempotent since it cannot be nilpotent. Yet as we have shown above, the ring $\mathbb{CFM}_{\mathbb{N}}(K)$ is not strongly π-regular. □

Therefore the class of strongly π-regular rings does not contain the class of INC rings.

Question 3.2.9. *Does there exist a strongly π-regular ring that is not INC?*

This remains an open question, as canonical examples of strongly π-regular rings known to the author have other convenient properties that imply INC.

CHAPTER 4

Nil Clean Rings and INC Rings

We are naturally motivated by the way we defined ideally nil clean, namely as a generalization of the nil clean property of elements in a ring, to resolve the following conjecture.

Conjecture 4.0.1. *If a ring R is nil clean, then it is ideally nil clean.*

This conjecture is motivated by the following observation about principal ideals of nil clean rings, which, in conjunction with Corollary 3.1.3 that a ring is INC if and only if every principal ideal is INC, allows us to characterize the relationship between a few classes of nil clean rings and INC rings.

Lemma 4.0.2. *Let I be an ideal of a ring R, and suppose that $a \in I$ is a nil clean element with nil clean decomposition $a = n + e$. Then the ideal I also contains n and e.*

Proof. Since $a \in I$ and $a = n + e$ where $n^k = 0$ for some $k > 0$ and $e^2 = e$, we observe that $na \in I$ where $na = n^2 + ne$ and that $a(1-e) \in I$ where $a(1-e) = n(1-e)$. It follows that $na + a(1-e) \in I$. We remark $na + a(1-e) = n^2 + ne + n(1-e) = n^2 + n = n(n+1) \in I$. Since we know that $1+n$ is a unit, we conclude $(1+n)^{-1}$ exists and $n(n+1)(n+1)^{-1} = n \in I$. It is immediate that $e \in I$ as well. □

Corollary 4.0.3. *For any element $a \in R$, if $a = n + e$ then $(a) = (n) + (e)$.*

Proof. Suppose $x \in (a)$. Then $x = \sum_{i=1}^{k} r_i a s_i$ for some $k > 0$. Moreover,

$$\sum_{i=1}^{k} r_i a s_i = \sum_{i=1}^{k} r_i n s_i + \sum_{i=1}^{k} r_i e s_i$$

so $x \in (n) + (e)$. And by lemma, $n \in (a)$ and $e \in (a)$ so if $x \in (n) + (e)$ then

$$x = \sum_{i=1}^{k} r_i n s_i + \sum_{i=1}^{m} r_i e s_i$$

for some $k, m > 0$ and all terms in both sums are already in (a), so $x \in (a)$. □

Corollary 4.0.4. *If R is a nil clean commutative ring, then R is ideally nil clean.*

Proof. In a commutative ring R we know the ideal generated by nilpotent elements is nil. Given a set of generators $S = \{a_i\}_{i \in \alpha}$ of ideal I, for each a_i let $a_i = n_i + e_i$ be a nil clean decomposition. Then we know that $S = \{e_i + n_i\}_{i \in \alpha}$. Consider the ideal generated by $\{e_i\}$, call this E, and the ideal generated by $\{n_i\}$, call it N. It is clear that $I \subseteq N + E$. To see that $N + E \subseteq I$, suppose $a \in N + E$. Then $a = \sum_{i,j \in I} n_i + e_j$. Since we assume $a_i, a_j \in I$ we know n_i and e_j are in I, thus $a \in I$. Therefore $I = N + E$ where N is a nil ideal and E is an idempotent ideal. □

In [2], Diesl defines a ring to be uniquely nil clean if every element of the ring has precisely one nil clean decomposition. He defines a ring to be strongly nil clean if every element a of the ring R has a nil clean decomposition $a = n + e$ such that $ae = ea$. His results allow us to demonstrate that these variations of nil clean coincide with INC.

Corollary 4.0.5. *If R is uniquely nil clean, then R is ideally nil clean.*

Proof. In [2], a consequence of Theorem 5.9 is that if R is uniquely nil clean then $R/J(R)$ is boolean, therefore every element is idempotent, which implies every ideal is idempotent in $R/J(R)$. Since every idempotents lift uniquely, the idempotent ideals in $R/J(R)$ lift to idempotent ideals in R, which means if I is an ideal in R then I is the sum of a nil ideal N contained in $J(R)$ and the ideal E generated by the idempotent lifts of the elements of $\bar{I}$, which we've shown to be idempotent by earlier results. So R is INC. □

In the case where R is a strongly nil clean ring, we can say more.

Theorem 4.0.6. *If R is a strongly nil clean ring then R is ideally nil clean.*

Proof. Suppose R is strongly nil clean and let I be any nonzero ideal of R. By Theorem 2.7 in [4], we know that $J(R)$ is nil and $R/J(R)$ is boolean. Therefore for all $x \in R$ we know that $\bar{x} = \bar{x}^2 \in R$. Since $J(R)$ is nil, we can lift $\bar{x}$ to an idempotent in R, call it e. Thus $x + n_1 = e + n_2$ where $n_1, n_2 \in J(R)$. Since n_1 and n_2 are in the radical, we know $(n) = (n_1 - n_2)$ is also a nil ideal. So $(x) = (e) + (n)$ is an INC decomposition. Hence, by Corollary 3.1.3 we know I is INC as well.

□

Corollary 4.0.7. *Suppose that for every idempotent ideal $I \subseteq R$ there exists a collection of idempotents that generates I. Then if R is INC, every ideal in R is generated by a collection of nil clean elements.*

The following lemma was suggested to us by A. J. Diesl, for whom it is named with much gratitude. We now supply a proof.

Diesl's Lemma. *If a is a strongly π-regular element of a ring R, then there exists a unit $u' \in U(R)$ such that $u'a$ is strongly nil clean.*

Proof. In [2] it is established that an element a of a ring R is strongly π-regular if and only if $a = e + u$ where e is an idempotent and u is a unit, and $ae = ea$, as well as eae is nilpotent. Since $a = ae + a(1-e)$ we may observe $u^{-1}a = u^{-1}ae + (u^{-1}e+1)(1-e) = u^{-1}ae + (1-e)$. Since a and e commute we know $u^{-1}ae$ is nilpotent. Therefore $u^{-1}a$ is strongly nil clean. □

Corollary 4.0.8. *If R is commutative, then every ideal generated by strongly π regular elements is INC.*

Proof. Suppose $a \in R$ is a strongly π-regular element. Then $(a) = (u'a)$ by Diesl's Lemma, and $(u'a)$ is INC by Corollary 4.0.3. Thus by Corollary 3.1.3 we conclude that any ideal generated by strongly π-regular elements is INC as well.

□

Theorem 4.0.9. *A commutative ring R is INC if and only if R is strongly π-regular.*

Proof. It is a fact from [6] that for a commutative ring R, we know R is strongly π-regular if and only if $R/J(R)$ is regular and $J(R)$ is nil. Therefore by Proposition 3.1.9 we know that the forward direction holds. For R being strongly π-regular implying that R is INC, observe that by Corollary 4.0.9, we know that all ideals in R are INC.

□

This has a few useful corollaries, due to established results concerning strongly π-regular rings.

Corollary 4.0.10. *If R is a commutative ring that is INC, then $J(R)$ is nil.*

This is merely a known fact about strongly π-regular rings.

Corollary 4.0.11. *If R is a commutative ring for which $R/J(R)$ is INC and $J(R)$ is nil, then R is INC.*

A final remark on the obstacles that remain in the way of characterizing rings that are both nil clean and ideally nil clean. In a noncommutative ring, it is quite possible that there are nilpotent elements which are not contained in any nil ideal. Ideals generated by such nilpotents, ones that are not in the Jacobson radical, are rather more complicated; if (n) is an ideal in a nil clean ring R, but not a nil ideal, then it contains an element b such that $b = m + f$ where m is a nilpotent element and $f^2 = f \neq 0$. Therefore $f \in (n)$ and if $n + f \neq n$ is nilpotent, it cannot be in $J(R)$, since $\bar{n} = \bar{f}$ is a contradiction, it would imply some power of f is in $J(R)$. In this case since $(n) = (n) + (f) = (n + f)$ there are multiple nilpotents not in the Jacobson radical that generate (n). If $n + f$ is not nilpotent, then by $(n) = (n + f)$ we know there is a non-nilpotent element of R that also generates (n).

Chapter 5

Variations of Ideally Nil Clean

5.1 Nilpotent Ideals and INC* Rings

The following definition and result are motivated by the question of whether artinian rings are ideally nil clean. If R is an artinian ring, then every nil ideal is nilpotent, and in fact $J(R)$ is nilpotent. As we will observe shortly, the hypothesis that an ideal is nilpotent rather than nil is far more restrictive for the ideal, and consequently allows us to say more about lifting elements of the quotient of R by the ideal in question.

Definition 5.1.1. *We say an ideal I is INC* if $I = N + E$ where E is an idempotent ideal and N is a nilpotent ideal.*

The following proposition capitalizes on how, unlike the sum of a nil ideal and another ideal, the nilpotent ideal in a sum disappears after taking a high enough power of the sum. The analog of this proposition for INC ideals, rather than INC*, remains an open conjecture as a result.

Proposition 5.1.2. *If $R/J(R)$ is INC* and $J(R)$ is a nilpotent ideal, then R is INC*.*

Proof. Since $J(R)$ nilpotent implies every $\bar{I}$ in $\bar{R} = R/J(R)$ is idempotent, $\bar{I}^2 = \bar{I}$ implies that for all $b \in I$ and $a \in I^2$, $b - a \in J(R)$. In this case, one need only consider the ideal in $J(R)$ generated by all $b - a$ to get a nilpotent ideal $\mathfrak{N}$ such that $I = \mathfrak{N} + I^2$. Recall also that $\bar{I}^2 = \bar{I}$ implies $\bar{I}^k = \bar{I}$and so $I = N + I^k$ for some nilpotent ideal $N \subseteq J(R)$ for all $k \geq 2$.

Observe that as a result, I is an idempotent ideal if and only if $\mathfrak{N} \subseteq I^2$. Otherwise, observe that if $n \in \mathbb{N}$ is the index of nilpotence of ideal $\mathfrak{N}$, then since $I = \mathfrak{N} + I^2$ for all $x_k \in I$ we know $x_k = \sum_{i=1}^{m} y_{k_i} z_{k_i} + b_k$ for some $y_{k_1}, z_{k_1}, \ldots, y_{k_m}, z_{k_m} \in I$, $m \in \mathbb{Z}_{\geq 1}$, and $b_k \in \mathfrak{N}$. Consider the set $\{x_k\}_{i=1}^{n}$ of any n elements of I, and observe that

$$\prod_{k=1}^{n}(x_k - \sum_{i=1}^{m} y_{k_i} z_{k_i}) = \prod_{k=1}^{n} b_k$$

Since $\mathfrak{N}^n = (0)$ we know $\prod_{k=1}^{n} b_k = 0$, and by expanding the left hand side, we see $x_1 x_2 \ldots x_n \in I^{n+1}$ since every other term in the expansion of the product is a product of at least $n+1$ elements of I. Thus $I^n \subseteq I^{n+1}$ which means I^n is an idempotent ideal. By above remarks, this means $I = I^n + N$ for a nilpotent ideal N. Thus R is INC*. □

Corollary 5.1.3. *Let R be a commutative ring. Then R is INC* if and only if $R/J(R)$ is regular and $J(R)$ is nilpotent.*

Proof. This follows from Corollary 4.0.9 and the above proposition. □

We now have the tools to confirm our intuition that there is a close relationship between artinian rings and INC rings.

Corollary 5.1.4. *Artinian rings are INC*.*

Proof. If R is artinian then $R/J(R)$ is semisimple. Moreover, $J(R)$ is nilpotent. Since a semisimple ring is the direct product of matrix rings over division rings, $R/J(R)$ can only have idempotent ideals. Thus by the above proposition, R is INC*. □

Since INC* implies INC, we obtain the following.

Corollary 5.1.5. *Any artinian ring is INC.*

This implies, among other things, that all finite rings are INC and all rings of the form $k[x]/(x^n)$ where k is a division ring and n is a positive integer are INC. However, we note that artinian rings are a proper subset of INC rings, since there are infinite direct products, such as $\prod_{i=1}^{\infty} \mathbb{Z}_2$, that are clearly INC but not artinian.

5.2 Ideally Nil Clean for One-Sided Ideals

We define what it means for a one-sided (left- or right-sided) ideal to be idempotent or nilpotent the same way we define idempotent and nilpotent two-sided ideals, respectively. In particular, if $\mathfrak{I} \subseteq R$ is an idempotent left ideal, then observe that $\mathfrak{I}^2 = \mathfrak{I}R\mathfrak{I}R$ is a left ideal that is contained in the two-sided ideal generated by the elements of $\mathfrak{I}$. The same holds for idempotent right ideals. Similarly if $\mathfrak{N} \subseteq R$ is a nilpotent left ideal then $\mathfrak{N}^k = (0)$ or $\mathfrak{N}(\mathfrak{N})^{k-1} = 0$ for some integer $k > 0$, where $(\mathfrak{N})$ is the two-sided ideal generated by the elements of $\mathfrak{N}$. The same holds for nilpotent right ideals.

Definition 5.2.1. *We say an ideal A in ring R is left ideally nil clean (right ideally nil clean) if there exist a nil left (right) ideal N and an idempotent left (right) ideal I such that $A = N + I$. We say R is LINC (RINC) if all ideals of R are LINC (RINC).*

Naturally, the first relationship we might hope to understand is that of LINC and RINC rings to INC rings. To that end, we make the following observations.

Remark 5.2.2. *Every LINC (RINC) ring is INC.*

Proof. Observe that if I is an ideal of a LINC ring R then I is also left ideal of R, so $I = N + E$ where N is a nil left ideal and E is an idempotent left ideal. Therefore $ER = RERER \subseteq I$ is an idempotent ideal and $NR = N$ is a nil ideal as a result. Similarly if R is instead an RINC ring. □

Lemma 5.2.3. *The sum of idempotent left (right) ideals is an idempotent left (right) ideal.*

Proof. Suppose $a \in \sum_{i\in\alpha} I_i$, meaning $a = r_1a_1 + \cdots + r_na_n$ where n is finite and $a_i \in I_i$. Since I_i is idempotent we know $a_i = b_ic_i$ for some $b_i, c_i \in I_i$. So a is the sum of elements of $(\sum_{i\in\alpha} I_i)^2$. □

We remark that not all INC rings are LINC or RINC. One ring that witnesses this distinction is Example 2.4.4 which we cited as an example of a ring with an idempotent ideal that is not generated by a collection of idempotents. It also serves as an example of a ring that is not regular, yet in which every ideal is idempotent [5].

Observe that in this ring the left ideal yxR is not itself idempotent, since $yxRyxR \subsetneq yxR$. It is also clearly not nil, since there are no nil (one sided) ideals in this ring. So this ideal must not be RINC. Similarly, it cannot be LINC.

The following are a few examples of ring that are both LINC and RINC.

Remark 5.2.4. *Every regular ring is both LINC and RINC.*

Proof. Since R is regular, for all $a \in R$ we know $a = axa$ afor some $x \in R$. Thus for all a we know $aR = axaR \subseteq aRaR$ and $Ra = Raxa \subseteq RaRa$. Since clearly $aRaR \subseteq aR$ and $RaRa \subseteq Ra$, we conclude that Ra and aR are idempotent (one-sided) ideals. Since the arbitrary sum of idempotent left (right) ideals is itself idempotent, any ideal $I \subseteq R$ is the sum of the principal left (right) ideals generated by its elements, therefore, I is LINC (RINC) as well. □

Example 5.2.5. *Any simple ring is both LINC and RINC.*

Proof. Suppose J is a left (right) ideal. Then since RJ is a two sided ideal, it is either (0) or R. Therefore, we know $JJ = JRJ = J(0)$ or $JR = J$. Therefore every left (right) ideal is either square nilpotent or is idempotent. □

In particular, observe that matrices over division rings are both LINC and RINC. We note that in a local ring, if $a \notin U(R)$ then $aR \subset J(R)$ is in a nil ideal, thus is nil. This leads us to the following example.

Example 5.2.6. *Local rings with a nil maximal ideal are LINC and RINC.*

This follows from the fact that the maximal ideal of a local ring is also maximal as a left ideal, and as a right ideal. So the Köthe conjecture is not an obstacle, and we conclude that all nontrivial one-sided ideals are nil.

Remark 5.2.7. *Artinian rings are LINC and RINC.*

Proof. Suppose that R is an artinian ring. Then every one-sided ideal is periodic, and every nil left (respectively, right) ideal is contained in $J(R)$, and therefore nilpotent. We know that $R/J(R)$ is semisimple, i.e. the direct product of simple rings. The left ideals of this ring are precisely the direct product of left ideals of the direct factors. Since the left ideals of simple rings are idempotent and therefore the left ideals of R are the sum of one such idempotent ideal in $R/J(R)$ and a nilpotent (left) ideal in $J(R)$. □

Example 5.2.8. *The ring of upper triangular matrices $\mathbb{T}_n(K)$ where K is a field is LINC and RINC.*

Proof. Since upper triangular matrices are artinian, this is an immediate corollary to the above remark.

□

In general though, we note that the Köthe conjecture is an obstacle to extensively characterizing LINC and RINC rings since unlike for INC rings, we only know that in general the sum of idempotent one-sided ideals is idempotent, but not the same for sums of nil one-sided ideals. If the Köthe conjecture holds, however, many of the results established for INC rings are true for LINC and RINC by entirely analogous proofs.

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