2014

Optimal Product Design for a Linear Pricing Monopolist

Sookyo Jeong
Wellesley College, sjeong@wellesley.edu

Optimal Product Design for a Linear Pricing Monopolist

by

Sookyo Jeong

Submitted to the Department of Economics
in partial fulfillment of the requirements for honors in

Bachelor of Arts in Economics

at

WELLESLEY COLLEGE

April 2014

Optimal Product Design for a Linear Pricing Monopolist

by

Sookyo Jeong

Submitted to the Department of Economics
on April 25, 2014, in partial fulfillment of the
requirements for the honors in
Bachelor of Arts in Economics

Abstract

This paper studies the optimal product design for a linear pricing monopolist. I ask what the profit maximizing strategy for a monopolist is, while making sure that it is correctly targeting at different types of consumers. Whenever perfect separation of the two types is informationally infeasible, the optimal menu of a linear pricing monopolist involves products that distort qualities for both types of consumers away from the first best allocations.

Thesis Supervisor: Casey G. Rothschild
Title: Diana Chapman Walsh Assistant Professor of Economics

Acknowledgments

My first and foremost gratitude goes to Prof. Casey Rothschild for the excellent advices and generous support during thesis, and throughout college. I am very lucky to have stumbled upon his guidance, without which my Wellesley experience would never have been the same. I could go on, but I will stop here. I would like to thank Prof. Eric Hilt, Yikang Li, Sophie Sun, Prof. Seth Neumuller, Prof. Courtney Coile, Prof. Olga Shurchkov, Prof. Akila Weerapana, the members of my thesis committee, and my fellow ERS classmates for the helpful comments.

My warm thanks to Ye Ji Kee, Jinyoung Byun, and Soo-jeong Lee–my chosen family–for enjoyable times, and for putting up with me through hard times. I also thank Guy-Manuel de Homem-Christo and Thomas Bangalter for their amazing music, which serves as a repository of randomly accessible memories of my college life.

I would like to express my sincere gratitude for my family for their love and support. Lastly, I send my prayer and condolences to the victims of Sewol ferry disaster, and their loved ones. May their souls rest in peace.

Contents

List of Figures

Chapter 1

Introduction

A monopolist facing buyers with heterogeneous tastes could sell a single product to all consumers, but the fact that different consumers get different surpluses out of the same goods indicates that there is a room for further surplus extraction. Instead of selling a single good, the monopolist can differentiate products to fully maximize its profits. Economists have been interested in this informational problem that arises when buyers have private information about their types while sellers do not. In particular, they have been investigating how the quality of the goods are affected when a seller cannot identify the type of buyer at the time of selling.

In this paper, I study the optimal product design for a linear pricing monopolist. I ask what the profit maximizing strategy for a monopolist is, while making sure that it is correctly targeting at different types of consumers. Price discrimination among buyers with private information is referred to as the second degree price discrimination, where the seller provides incentive for the buyers to differentiate among themselves. I demonstrate in this paper that, when any type of buyer has a binding incentive compatibility constraint, an optimal menu of a linear pricing monopolist involves products that distort qualities for both types of consumers away from the first best allocations, or the optimal allocations in the absence of informational asymmetry.

While there is an extensive body of literature that deal with optimal product design for nonlinear pricing monopolists, there is a gap in the literature that studies product design linear pricing monopolists. My thesis attempts to fill this gap, by looking at the quality distortions involved when the monopolist is forced to price linearly. The primary reason that the second degree price discrimination in linear pricing case has not been addressed is that, when there is only a single dimension on which the goods can be differentiated, linear pricing mechanically precludes product differentiation. (Stiglitz, 1977) But in the light of the fact that most goods have multiple quality dimensions, this paper reintroduces the possibility of using product differentiation to target distinct types.

The optimal product design for a linear pricing monopolist is going to involve quality distortions, just as with nonlinear pricing monopolist case, but with both types getting inefficient allocations when the incentive compatibility of the either type binds. I first explain the difference between nonlinear pricing monopolist and linear pricing monopolist, by explaining market constraints that forces monopolists to price linearly, and then approach the more general profit maximization problem of a monopolist with second-degree price discrimination, by addressing the two main aspects: incentive compatibility, and profit maximization.

First, let us investigate the market constraints that results in the difference between non-linear pricing monopolist, and linear pricing monopolist. A monopolist is forced to price linearly, if it can neither (i) prevent resale, nor (ii) observe the total amount purchased by a buyer. For example, consider a monopolist selling cereals to two types of buyers in the market: buyers with high preference for sugar, and others with high preference for fiber. The monopolist cannot provide bulk discounts since if it did, a single buyer would purchase all cereals, and sell them at higher prices in a secondary market. The monopolist can neither charge more for the last few units, since consumers can buy the same cereal at different stores, or come back and buy the same cereal multiple times without being noticed by the monopolist. Therefore, the monopolist, when facing secondary market and non-exclusivity, is forced to price linearly.

Naturally, since linear pricing case involves more mathematical restriction than nonlinear pricing case, it is convenient to start with nonlinear pricing case to investigate the source of informational rent and quality distortion, which are two direct consequences of informational asymmetry in the second degree price discrimination.

A natural example of nonlinear pricing monopolist would be a monopolist selling airplane tickets to two types of buyers: rich businessmen, and bargain hunter vacationers. Businessmen have higher income, and higher willingness to pay for a better quality seat. Bargain hunter vacationers are willing to forego on seat comforts to save extra dollars. The monopolist would be able to price nonlinearly, because it can prevent resale among buyers by printing the name of the passenger on the ticket, and because it can observe the total amount of tickets purchased by a single buyer.

The optimal profit maximizing strategy for a nonlinear pricing monopolist engaging in the second degree price discrimination still involves constraints, namely: (i) incentive compatibility constraint, and (ii) participation constraint. Incentive compatibility ensures that a buyer does not have any incentive to deviate from the product targeted at her; i.e., that the product targeted at her gives her higher level of utility than the other products that are offered in the market. Participation constraint ensures that buyers do not leave the market; that is, the product targeted at her gives higher level of utility than the utility level she can get from outside the market. We call a set of products that are offered in the market as 'a menu' and say that a menu is incentive compatible if every allocation on the menu is such that the targeted consumer does not deviate, and correctly consumes that allocation.

The optimal strategy for the airplane ticket monopolist is to design a menu that consists of first-class seats, targeted at the businessmen, and coach seats, targeted at the vacationers. For such menu to be incentive compatible, the monopolist has to make sure that businessmen are actually buying first class seats, and not bearing the low quality coach seats to save spare hundreds of dollars to spend on other things. In order for the menu to satisfy participation constraints, the monopolist also has to make sure that both businessmen and bargain hunter vacationers are getting the levels of utility that they would have gotten from outside the market.

In order to fully motivate the study, and put it into a context, I start by investigating three different types of markets in the paper before moving onto the fourth market, which illustrates the main point of my thesis. In chapter 3, I consider a market where monopolist has perfect information. In chapter 4.1, I consider a market with nonlinear pricing monopolist providing single dimensional goods. In chapter 4.2, I consider a market with nonlinear pricing monopolist providing multidimensional goods. Lastly, in chapter 5, after explaining the set up and delivering the conventional wisdom in the screening mechanism literature, I turn to the fourth market where the linear pricing monopolist provides multidimensional goods to two types of buyers.

The first market, where the monopolist has perfect information, entails standard results from the principles of economics–more specifically, its efficient allocations involve maximizing social surplus. Here we derive the notion of the first-best allocations, which we use as benchmark allocations to see whether the allocations with informational asymmetry are distorted or not.

The second market, where nonlinear pricing monopolist provides single dimensional goods to two types of buyers, follows the model of Mussa and Rosen

(1978), and is included to illustrate the source of informational rent and quantity distortion. Specifically, I use this model to demonstrate that the buyer with higher valuation of the good gets informational rent with first-best allocation, while the buyer with lower valuation of the good gets allocations distorted away from the first-best.

The third market, where nonlinear pricing monopolist provides multi-dimensional goods to two types of buyers, illustrates the point that even if the monopolist can differentiate goods based on multiple attributes, the standard result from the single dimensional case persists– that is, the buyer with the highest valuation of the good gets the first-best allocations.

In the last market, where a linear pricing monopolist provides multidimensional goods to two types of buyers, we show that there is an additional distortion that has not been discussed with the linear pricing monopolists, that arises from the ability of the buyers to buy some of both products in the market without being observed by the monopolist. I introduce a new type of incentive compatibility constraint from Rothschild (2014), which is the linear analogue to the standard incentive compatibility constraint, and which correctly captures the market constraint of the linear pricing monopolist.

Chapter 2

Set Up

In this chapter, we lay out the set up and assumptions of the problem. The monopolist faces two types of buyers in the market, H and L. The H type consumers have higher willingness to pay such that the single crossing property holds. The monopolist can design two products targeted at each type. If the monopolist can price nonlinearly, the revelation principle applies (Myerson, 1979), and it is sufficient for a profit maximizing monopolist to provide two goods targeted at each type, $Q_H = (q^H, p^H)$ and $Q_L = (q^L, p^L)$. If the goods are differentiated on two dimensions, two goods are represented by $Q_H = (q_1^H, q_2^H, p_H)$, and $Q_L = (q_1^L, q_2^L, p^L)$. If the monopolist is forced to price linearly, however, the standard revelation principle is undermined, since the buyers can freely take linear combinations of the contracts. Nevertheless, we can still refer to the allocations targeted at each type as $Q_H = (q_1^H, q_2^H, p_H)$, and $Q_L = (q_1^L, q_2^L, p^L)$, and think about when the contracts would be incentive compatible.

Consumers have preferences represented by the utility function of CRRA class, and the heterogeneous tastes of different types of buyers are represented by the parameters b_1 and b_2 as below:

$$\begin{aligned} U_i(q_1,q_2) &= u(b_1^i q_1) + u(b_2^i q_2) - p_i \\ &= \frac{(b_1^i q_1)^{1-\gamma}}{1-\gamma} + \frac{(b_2^i q_2)^{1-\gamma}}{1-\gamma} - p_i \qquad i \in \{H,L\} \end{aligned} \tag{2.1}$$

Note that U is continuously differentiable, strictly concave, and strictly increasing in q_1 and q_2. $b_H > b_L$, so the H type will get higher utility than the L type for each additional unit of the good; i.e., the H type will have steeper utility curve than the L type. We also assume that $\gamma > 1$. $0 < \lambda < 1$ is a proportion of H type consumers in the market.

We note that although we use the CRRA class of utility function for explicit exposition and algebraic simplicity of the problem, the main result of the paper persists as long as preferences are homothetic. (See Lemma 10.) That is, if we are willing to believe that a buyer demands the same proportion of attributes given different prices, the result persists even if the ratio of marginal utility with respect to the attributes of the goods is not constant.

The monopolist wants to maximize profits

$$\max_{Q_H,Q_L} \Pi(Q_H,Q_L) = \lambda\left[p_H - C(Q_H)\right] + (1-\lambda)\left[p_L - C(Q_L)\right]., \quad i \in \{H,L\} \tag{2.2}$$

The monopolist faces constraints when maximizing profits: it has to make sure that it (1) provides as much utility level as the outside market options, and (2) correctly targets the products at each type. We assume throughout that there are 'participation constraints' that the utility provided by the i types' allocation to i types must exceed the minimum utility $\overline{U}_i$. Additionally, the monopolist will potentially face informational "incentive" constraints when it cannot observe the type of buyers at the time of selling. The nature of these incentive constraints is context dependent, and we discuss them further below.

We assume the cost function $C(q_1^i, q_2^i) = c_1 q_1^i + c_2 q_2^i$ to be linear in q_1^i and q_2^i, for $i \in \{H, L\}$. It is worth noting that we only require linearity in the presence of the multiple types of buyers in the market. With more than one type of buyers, the additivity of the cost function matters, as well as homogenous of degree 1 property. However, we note that if there is a single type of buyers in the market, additivity becomes a trivial issue, and we can regard the cost function to be of more general form with homogenous of degree 1 property.

$$C(q_1, q_2) = ((q_1)^\alpha + (q_2)^\alpha)^{1/\alpha}$$

For the rest of the paper, however, we assume linear cost function even for the case with single type of buyer, for simplicity and comparability across different settings.

Chapter 3

Allocations Without Informational Asymmetry

In this chapter, we look at a market where the monopolist has perfect information–that is, it can (1) observe the type of buyer at the time of selling, and can (2) observe the total amount purchased by each buyer. Because the market involves no informational asymmetry between the monopolist and the buyers, optimal (profit maximizing) allocations are referred to as the "first best" allocations. In this market, the buyers get allocations that they would have gotten in the absence of the other type. In other words, the presence of other types of buyers in the market do not affect the quality of their goods, since the monopolist can correctly identify the types of buyers at the time of selling. The first best allocations, or the 'undistorted' allocations, are baseline cases against which we compare the 'distorted' allocations in the presence of informational asymmetry in the next chapters.

We first start by looking at the monopolist selling two goods targeted at two consumers, H, and L. We assume, for simplicity, that arbitrage or resale is not possible among buyers, so that monopolists can price nonlinearly. Note that these 'nonlinear pricing assumptions' are naturally satisfied in the case of perfect information. The monopolist thus solves profit maximization problem with four choice variables $(q_H, p_H), (q_L, p_L)$, where each consumer gets a quantity q at a price p. The set of the problem follows (2.2), except for the absence of the incentive compatibility constraints, since the monopolist can identify the type of buyers at the time of selling in this chapter.

$$\max_{\{(q_H,p_H),(q_L,p_L)\}} \lambda (p_H - C(q_H)) + (1-\lambda)(p_L - C(q_L)) \tag{3.1}$$
$$\text{subject to } U_H(q_H, p_H) - \bar{U}_H \geq 0$$
$$U_L(q_L, p_L) - \bar{U}_L \geq 0$$

$C(q_1, q_2) = c_1 q_1 + c_2 q_2$ is linear, thus weakly convex, and $U(q_1, q_2)$ is given by (2.1).

$$\begin{aligned} U_i(q_1, q_2) &= u(b_1 q_1) + u(b_2 q_2) - p_i \\ &= \frac{(b_1^i q_1)^{1-\gamma}}{1-\gamma} + \frac{(b_2^i q_2)^{1-\gamma}}{1-\gamma} - p_i \qquad i \in \{H, L\} \end{aligned} \tag{3.2}$$

where U is continuously differentiable, strictly concave, and strictly increasing in q_1 and q_2. $0 < \lambda < 1$ is a proportion of H type consumers in the market. $b_H > b_L$, so the H type will get higher utility than the L type for each additional unit of the good; i.e., the H type will have steeper utility curve than the L type.

That the monopolist maximizes the above object function using four choice variable implies that there are four first order conditions associated with the following Lagrangian:

$$\mathcal{L} = \lambda \Pi_H(q_H, p_H) + (1-\lambda)\Pi_L(q_L, p_L) + \mu_1 [U_H(p_H, q_H) - \bar{U}_H] + \mu_2 [U_L(p_L, q_L) - \bar{U}_L] \tag{3.3}$$

$$\text{FOC's: } \lambda\frac{\partial\Pi_(p_H,q_H)}{\partial q_H}+\mu_1\frac{\partial U_H(p_H,q_H)}{\partial q_H}=0$$
$$\lambda\frac{\partial\Pi_(p_H,q_H)}{\partial p_H}+\mu_1\frac{\partial U_H(p_H,q_H)}{\partial p_H}=0$$
$$(1-\lambda)\frac{\partial\Pi_(p_L,q_L)}{\partial q_L}+\mu_2\frac{\partial U_L(p_L,q_L)}{\partial q_L}=0$$
$$(1-\lambda)\frac{\partial\Pi_(p_L,q_L)}{\partial p_L}+\mu_2\frac{\partial U_L(p_L,q_L)}{\partial p_L}=0$$

Solving for μ_1 and μ_2, and rearranging the expressions, the above condition reduces down to

$$\frac{\partial U_H(p_H,q_H)/\partial p_H}{\partial U_H(p_H,q_H)/\partial q_H}=\frac{\partial\Pi(p_H,q_H)/\partial p_H}{\partial\Pi(p_H,q_H)/\partial q_H}, \text{ and}$$
$$\frac{\partial U_L(p_L,q_L)/\partial p_L}{\partial U_L(p_L,q_L)/\partial q_L}=\frac{\partial\Pi(p_L,q_L)/\partial p_L}{\partial\Pi(p_L,q_L)/\partial q_L}$$

These two conditions tell us that, in a profit maximizing allocation, the marginal rate of substitution should be equal to the marginal rate of transformation. Graphically, this means that at a first best allocation, utility curve is tangent to the iso-profit curve on a (q,p) plane.

Now we derive heuristically, by similar reasoning, that a profit maximizing allocation necessarily satisfies $\frac{\partial\Pi(p_H,q_H)/\partial p_H}{\partial\Pi(p_H,q_H)/\partial q_H}=\frac{\partial\Pi(p_L,q_L)/\partial p_L}{\partial\Pi(p_L,q_L)/\partial q_L}$. This condition has to do with the monopolist deciding the proportion of resources to put in each type's production to maximize profits. Suppose, by way of contradiction, that at a profit maximizing allocation, $\frac{\partial\Pi(p_H,q_H)/\partial p_H}{\partial\Pi(p_H,q_H)/\partial q_H}>\frac{\partial\Pi(p_L,q_L)/\partial p_L}{\partial\Pi(p_L,q_L)/\partial q_L}$. This means that by serving more of H type and less of L type, monopolists would be able to increase profits. But by doing so, $\frac{\partial\Pi(p_H,q_H)/\partial p_H}{\partial\Pi(p_H,q_H)/\partial q_H}$ decreases, and $\frac{\partial\Pi(p_L,q_L)/\partial p_L}{\partial\Pi(p_L,q_L)/\partial q_L}$ increases. There is a contradiction, proving that this inequality cannot hold in profit maximizing allocations. Graphically, this means that the slope at which H type's allocation is tangent to its utility curve and iso-profit curve must be identical to the slope at which L type's allocation is tangent to its utility curve and iso-profit curve.

In this chapter, we saw two important properties of the first best allocations: it is where (i) the utility curve is tangent to the iso-profit curve on a (q, p) plane, and where (ii) the slopes of the tangents for both types are identical. Figure 3-1 shows an example of such allocation. The pink iso-cost lines $\overline{\Pi}_i, i \in \{H, L\}$ are tangent at the utility curves of each type. We note that the slopes of the iso-cost lines for both types are identical.

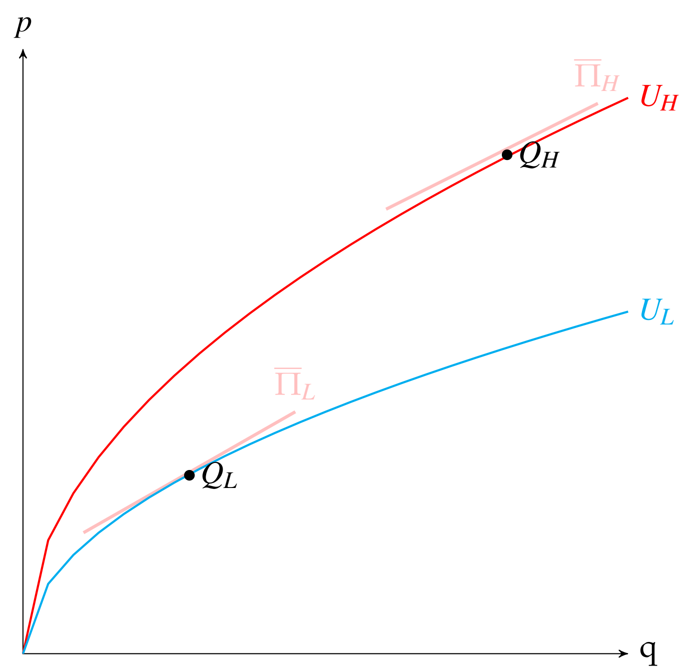


Figure 3-1: First best allocation example

As a closing note, and to motivate the next chapter, I briefly discuss how the existence of informational asymmetry might affect the optimal allocations. Note that by implementing the allocations $\{Q_H, Q_L\}$ in Figure 3-1, the monopolist fully extracts surpluses from both types, thus maximizing its profit. This implementation is possible with perfect information– that is, if the monopolist can identify the types of buyers at the time of selling. However, if the types of buyers are private information not known to the monopolist, then such allocation is not implementable, specifically because it is not incentive compatible. The H type can do better by consuming Q_L instead of Q_H: it gives them higher utility level. Therefore, in order to devise an optimal strategy, the monopolist now has to take incentive compatibility into account, and make sure that the buyers do not have incentive to deviate from the products targeted at them. In the next chapter, I carry the assumptions of the nonlinear pricing monopolist selling single dimensional good, where it can prevent resale, and where sales are exclusive. I add in the incentive compatibility constraint that reflects informational asymmetry between principal and agent, and discuss the two main characteristics of the optimal menu that are direct consequences of informational asymmetry: informational rent given to the H type, and quality distortion involved in the L type's product.

Chapter 4

Literature Review: Nonlinear Pricing

4.1 Nonlinear Pricing with Single Dimensional Good

In this section, we consider a class of problem involving the monopolist selling single dimensional goods to two types of buyers, in a market where (1) resale is preventable, and where (2) the monopolist can observe the total amount of goods purchased. We follow the canonical one dimensional nonlinear pricing monopolist model of Mussa and Rosen (1978), while simplifying the types to be discrete rather than continuous, and using quantity, instead of quality as the attribute that the buyers care about. We also adopt CRRA class of utility function instead of linear utility function, in order to make the results across the cases considered in this paper more comparable. The modifications do not substantively impact the key results of interest, that while the buyer with higher valuation on the good gets the efficient first-best allocation, the buyer with lower valuation on the good gets an allocation distorted away from the first-best.

If the monopolist operates in nonlinear pricing market, where sales are exclusive and where it cannot prevent resale, it can make take-it-or-leave-it offer to buyers, such that the consumers' utility maximization problem reduces down to either taking the product if it gives them higher utility than an outside option, or leaving it for an outside option. The revelation principle (Myerson, 1979) applies, and we can without loss of generality consider directly assigning allocations to the two types as long as they are incentive compatible and better than the outside option. The monopolist designs a menu of products $\{(q_H, p_H), (q_L, p_L)\}$ that would (i) make the consumers be willing to participate in the market, and (ii) make different types of consumers to self-select themselves into targeted products. The optimal product design that would maximize profits and successfully price discriminate between different types of consumers would be the solution to the following problem.

$$\max_{p_H, p_L, q_H, q_L} \lambda(p_H - C(q_H)) + (1-\lambda)(p_L - C(q_L)) \tag{4.1}$$

$$\text{s.t. } U_H(q_H, p_H) \geq \overline{U_H} \tag{MU_H}$$

$$U_L(q_L, p_L) \geq \overline{U_L} \tag{MU_L}$$

$$U_H(q_H, p_H) - U_H(q_L, p_L) \geq 0 \tag{IC_H}$$

$$U_L(q_L, p_L) - U_L(q_H, p_H) \geq 0 \tag{IC_L}$$

As with before, C is weakly convex, and $U_i(p,q) = u_i(q) - p$, where $u_i(q) = \frac{b_i q^{1-\gamma}}{1-\gamma}$ of CRRA form is strictly concave, given by (2.1). $b_H > b_L$, so the H type will get higher utility than the L type for each additional unit of the good. $0 < \lambda < 1$ is a proportion of H type consumers in the market.

The minimum utility constraints MU_H and MU_L ensure full participation of the H type and the L type consumers respectively, where the right hand side of the constraints refer to the level of utility that the buyers can get by leaving the market. The incentive compatibility constraints IC_H and IC_L ensure perfect competitive screening such that no buyer is willing to deviate to a product other than her own.

Let us now turn to the graphical framework. (See Figure 4-1.) The x-axis represents quantity, and the y-axis represents both utility that consumer gets and the price that consumer pays. The fact that y-axis represents both utility level and price reflects the fact that the monopolist fully extracts consumer surplus when it engages in perfect screening: if the price is lower than the utility, the monopolist can always raise the price until the minimum utility constraint binds.

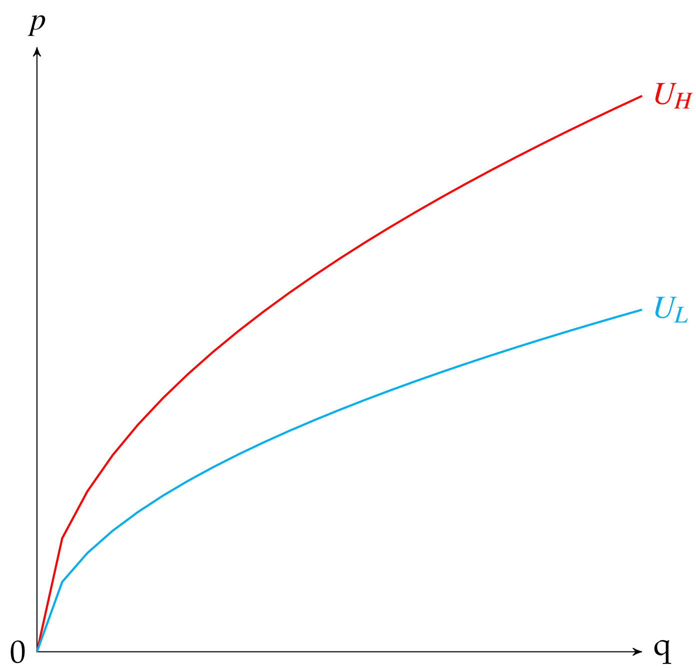


Figure 4-1: Preferences

Before moving on, it is worth noting two things about the way the graph is drawn. First, consumers get higher utility as the allocations move to lower right. Secondly, as a corollary, feasible allocations that satisfy minimum utility constraints will lie lower right to the minimum utility curve.

Let us look at what each of the constraints in the profit maximization problem refers to. We first start with the minimum utility constraints, MU_H and MU_L. The aforementioned implication of minimum utility, that it is the borderline between buying and not buying the product, indeed calls for two notable corollaries: (i) that it is always going to be the case that one of the minimum utility binds, and (ii) that any type whose minimum utility does not bind gets *informational rent*, in the presence of informational asymmetry. To illustrate the first point, let us suppose, by way of contradiction, that neither of the minimum utility constraints bind. Then the monopolist would have incentive to increase the both types' prices by the same amount without changing quantities. This movement increases profits, and is feasible when both MU's are slack, since the incentive compatibility constraints are still satisfied. Consumers are going to take the product as long as the allocation lies to the bottom right of the minimum utility curve. [1]Thus, the monopolist can raise price until one of the minimum utility constraint binds.

We now consider the second point. With nonlinear-pricing monopolists, the graphical representation of incentive compatibility constraint is going to be the indifference curve on which the targeted allocation lies. The H type incentive compatibility constraint says that Q_L must lie above and to the left of H's indifference curve through Q_H. Figure 4-2 represents feasible allocations $\{Q_H, Q_L\}$ such that the allocations lie on each type's minimum utility curves. Given these allocations, the incentive compatibility constraint of the H type is U_H, and that of the L type is U_L. We note, however, that the allocation Q_L lies to the bottom right of H type's indifference curve. What this means is that the H type gets more utility from buying Q_L than buying Q_H, or equivalently, $U_H(q_H, p_H) - U_H(q_L, p_L) < 0$, thus violating incentive compatibility constraint IC_H. We see that, by considering the utility curve of the H type that goes through Q_L, the H type gets utility level of U'_H by buying Q_L, and gets U_H by buying Q_H. Clearly, $U'_H > U_H$, so the H type has incentive to deviate from its targeted product when presented with a menu $\{Q_H, Q_L\}$. In fact, closer inspection gives that H type consumer would always have incentive to deviate *unless*

[1]One might wonder, why the monopolist would not provide allocations such that both of the minimum utility curves bind. After all, it seems like the most effective way to fully extract consumer surplus, thus maximizing profit. It turns out that such strategy always turns out to be incentive incompatible. In the end, if only one type's minimum utility binds, the other type ends up getting informational rent, which is a rent that she derives from having private information that is not revealed to the monopolist. In order to price discriminate the two types, the monopolist would have to provide this type with sufficient rent so that she does not have any incentive to deviate from her own product.

H type's allocation lies on U'_H.

We have now established the second point, that any type whose minimum utility does not bind gets informational rent. In other words, informational asymmetry prevents the monopolist from fully extracting each type's reserve utilities when they are different. ($\overline{U}_H \neq \overline{U}_L$.) If we assume that $\overline{U}_H < \overline{U}_L$, then it is always going to be the case that minimum utility constraint of the L type consumers binds, while that of the H type consumers is slack. Intuitively, this means that the L type consumers are always going to be indifferent between buying and not buying the product, while the H type consumers are going to be strictly better off by buying the product. In fact, we can quantify the exact amount by which the H type consumers are going to be better off, which is $U'_H - U_H$. To see the implication more clearly with figures, we suppose that the monopolist fixes quantity and only adjusts price. Then the amount by which H type is going to be better off, or *the informational rent*, can be represented by the length of the green line (See Figure 4-2), or equivalently, the amount by which the price decreased for the same quantity. In other words, informational rent is something that monopolists provide to the H type consumers such that they do not have incentive to deviate from their targeted products.

Now that we have discussed the implications of participation constraint and incentive compatibility constraint, we turn to the profit maximization problem of the monopolist. We ask: how does the monopolist pin down the optimal allocation of q's and p's? After all, even if we do impose minimum utility constraints and incentive compatibility constraints, the monopolist still faces infinite amount of bundles on the iso-utility curves to choose from. The standard result from the principles of economics, of course, implies that the monopolist sets marginal revenue equal to the marginal cost. That is, they choose bundles on the indifference curves that are tangent to their iso-profit curves. Iso-profit curves connect allocations that give the

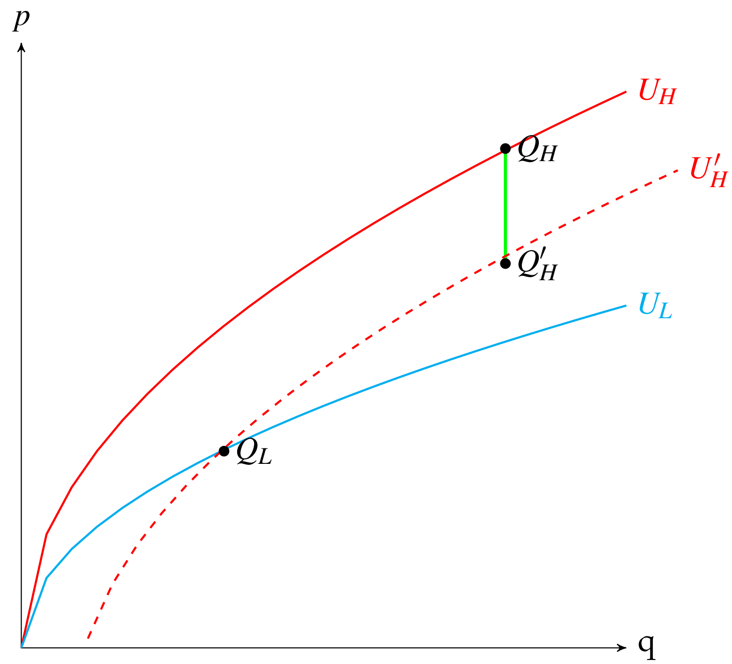


Figure 4-2: informational rent in green

monopolist the same amount of profits. In other words, if the monopolist moves allocations by adjusting p's and q's such that the resulting allocations give exactly the same amount of profits as before, the old allocations and the new allocations must lie on the same iso-profit curve. From the previous chapter, we saw that the first best allocations lie on the tangency of the utility curve and the iso-profit curve. However, in the presence of informational asymmetry, the first best allocations are not going to be optimal.

Specifically, we are going to look at how profit maximization involves distortion in quality for one, and only one type of buyer. Consider allocations $\{Q_H, Q_L\}$ for each type, such that they are first best for both types, and such that they satisfy both incentive compatibility constraints and participation constraints. The monopolist can always do better by moving L's allocation away from the first best, thus creating "distortion". In Figure 4-3, moving the point labeled Q_L down and to the left along L's indifference curve to Q''_L has two effects: (1) it lowers profits from the L types, and (2) it eases the binding H type incentive constraint. Since the H

type earned positive rents (i.e., has a non-binding minimum utility constraint) at the original allocation, the second effect allows the monopolist to raise the price of the H type contract without violating incentive compatibility, which raises profits from the H type. So there are two profit effects: the direct effect from (1), which decreases profits on L types, and the effect from (2), which increases profits on H types. The key observation is that, for small distortions in L, the second effect is larger. Intuitively, this is because, by assumption, allocation L was "first best" and thus locally profit maximizing. So a movement of size ε only effects profits on L to second order in ε. But because the H type indifference curve through L is steeper, this same movement eases the H type incentive constraint to first order in ε, so the profit from the second effect is first order in ε. The exact amount of surplus by which monopolist additionally extracts by taking such action is represented by the blue line in Figure 4-3.

We conclude the section by looking at the example of the monopolist selling airplane tickets, and see intuitively how the results apply. The monopolist faces two types of buyers in the market: rich businessmen who cares more about comfort than money, and thrift bargain hunter vacationers who would forego seat comfort to save on dollars. The optimal strategy of the monopolist involves designing two products targeted at each type–first class seats for the businessmen, and coach seats for the bargain hunter vacationers. The problem that the monopolist faces from such product differentiation is that businessmen might have incentive to deviate from first class seats, and bear through the coach seats to save spare hundreds of dollars to spend on other things. In order to prevent this from happening, monopolist would deliberately distort the quality of the coach seats down to scare businessmen away from buying coach seats. Monopolist can then raise the price of the first class seats further, thus fully extracting the surplus of the businessmen generated by the quality distortion.

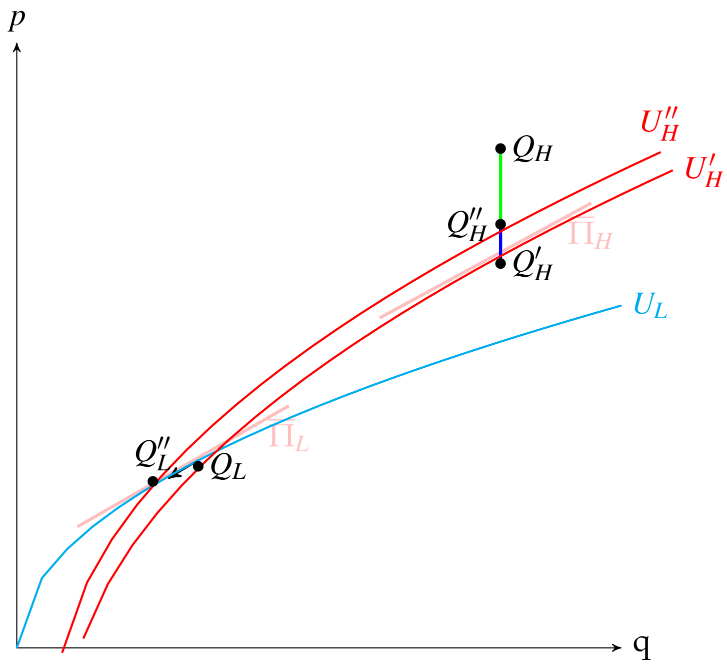


Figure 4-3: Increase in Profit by Distortion

4.2 Nonlinear Pricing with Two Dimensional Good

We have seen the sources of informational rent and quality distortion involved in the optimal product design for a nonlinear pricing monopolist providing single dimensional goods in the previous section. In this section, we look at the nonlinear pricing monopolist's products when the goods have multiple attributes. The main purpose of presenting the nonlinear pricing case is to compare and contrast it with the linear pricing case in the next chapter. To do so, we extend the previous single dimensional model, and lay out the exposition in the simplest way possible. The results from canonical nonlinear multidimensional screening literatures suggest that the optimal strategy for the nonlinear pricing monopolist providing multidimensional goods involves only one type who puts the highest value on the good gets the undistorted, first-best allocation. (Armstrong and Rochet, 1999) In multidimensional screening, there will be a new dimension of distortion. The purpose of this section is not to pin down the optimal product allocation in the nonlinear pricing case, but to demonstrate that, as in the single dimensional case, no distortion on at least one boundary of the type space persists.

To motivate this section, consider the monopolist selling airplane tickets to rich businessmen, and thrift bargain hunter vacationers, as in the previous section. Although in the previous section, we considered overall comfort of the seats as the attribute that measures quality of the product, we can also see that quality, instead of being a one-dimensional thing, could be thought of as consisting of two things that buyers might care about. For example, flyers might care about food and drink quality as well as comfort of seats. And it could be the case that bargain hunter vacationers care more about food quality, and businessmen might care less about food quality, because they are sleep deprived, and are going to sleep through the flight anyway. This means that to satisfy incentive compatibility, the monopolist can distort both dimensions of quality, and we intuitively expect the monopolist to distort seat comfort more than it distorts food quality, since the degradation of seat comfort dissuades businessmen more effectively.

It turns out that the optimal strategy involves the monopolist deliberately degrading the seat comfort while increasing food quality by a little bit for the vacationers, and serving the first-best quality to the businessmen. Let us see intuitively how this is so. We illustrate the intuition with Figure 4-4. The x-axis represents food and drink quality, and the y-axis represents seat comfort. We assume that both types get allocations of the same price, to omit the price axis for simplicity. As we will see later, this assumption does not affect the main result. U_B is the indifference curve of the businessmen and likewise, U_V is the indifference curve of the vacationers. The pink rays, r_B and r_V, that go through the origin are 'undistorted rays' on which the first best allocations lie for each type. Profit is maximized on these rays for each utility level. We prove the mathematics for the graph shortly after illustrating the intuition with this example.

Suppose first, that there are only vacationers, and no businessmen in the mar-

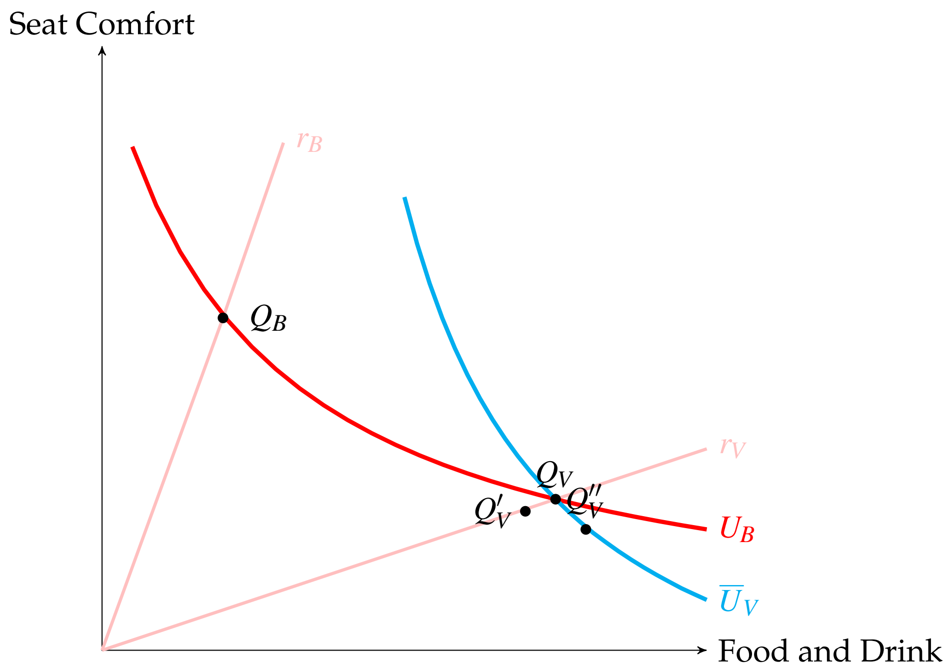


Figure 4-4: Airline monopoly with two dimensional good.

ket. Then the monopolist would be able to correctly identify the type of buyer at the time of selling, and the optimal allocation would be the first-best allocation, lying on the straight undistorted ray. The monopolist would be providing seat comfort, and food and drink quality that would make vacationers just as willing to buy tickets, thus providing the minimum utility level. Now, suppose that there are also businessmen in the market, in addition to the vacationers. Businessmen have their own undistorted seat design ray r_B, which is comprised of allocations that maximizes monopolist's profits, just as in vacationers' undistorted ray r_V. The ray for businessmen is steeper than that of vacationers, since businessmen care more about seat comfort than food and drink quality. If the monopolist can correctly identify the type of buyer at the time of selling, then it could sell Q_B and Q_V, which are on the undistorted rays of each type, since it's where the profits are maximized. However, since coach seats are way cheaper, businessmen might bear through coach seats to spend spare hundreds of dollars on other things. And since the monopolist cannot observe the type of buyer at the time of selling, it faces the problem that businessmen might deviate from the first-class seats and buy cheaper coach seats. So the monopolist wants to make coach seats less desirable, but it doesn't want to make coach seats more expensive, while holding quality constant, because it might prevent vacationers from entering the market at all. So, instead, it wants to lower the quality of coach seats.

Suppose that the monopolist reduces both seat comfort and food and drink quality for coach seats, from Q_V to Q'_V. This movement would of course prevent businessmen from buying coach seats, but it can also prevent vacationers from buying coach seats since they are getting lower utility per dollar than their minimum level. So to make sure that businessmen are not buying coach seats, and that vacationers are not leaving market, the monopolist would decrease seat comfort of coach seats, while increasing food and drink quality by a little bit. ($Q_V \rightarrow Q''_V$ in Figure 4-4.) By doing so, it can provide the same level of utility per dollar for vacationers, thus making sure that vacationers do not leave the market; also, it prevents businessmen from buying coach seats. Increase in food drink quality for coach seats would not be able to compensate for the decrease in seat comfort, since businessmen care more about seat comfort than food and drink quality. So indeed, we get distorted quality for one and only one type of buyer, in this case, coach seats for vacationers.

Let us abstract away from the example, and prove the intuition with mathematics. In the nonlinear pricing case, monopolists can still prevent resale and observe the total amount of goods purchased, and therefore can make a take-it-or-leave-it offers to consumers. The only feature that is different from the previous section is the multi-dimensionality of the products. What this means is that the monopolist now has three control variables–price, first characteristic, and the second characteristic–as opposed to two that it had before. The problem could be stated formally as following:

$$\max_{p_H, q_1^H, q_2^H, p_L, q_1^L, q_2^L} \lambda(p_H - C(q_1^H, q_2^H)) + (1-\lambda)(p_L - C(q_1^L, q_2^L)) \tag{4.2}$$

$$\text{s.t. } U_H(q_1^H, q_2^H, p_H) \geq \overline{U_H} \tag{MU_H}$$

$$U_L(q_1^L, q_2^L, p_L) \geq \overline{U_L} \tag{MU_L}$$

$$U_H(q_1^H, q_2^H, p_H) - U_H(q_1^L, q_2^L, p_L) \geq 0 \tag{IC_H}$$

$$U_L(q_1^L, q_2^L, p_L) - U_L(q_1^H, q_2^H, p_H) \geq 0 \tag{IC_L}$$

$\lambda \in (0,1)$ is the proportion of H type consumers in the population, known to the monopolist. The minimum utility constraints MU_H and MU_L ensure full participation of the H type and the L type consumers respectively, and the incentive compatibility constraints IC_H and IC_L ensure perfect competitive screening such that no buyer is willing to deviate to a product other than her own.

Intuitively, screening is possible in this case with similar results as before. More specifically, the L type consumers, whose minimum utility constraint binds, are going to have an allocation distorted away from the first best efficient allocation, and the H type consumers are going to get their first best allocation.

To illustrate the result, we develop a graphical framework for the setting. (See Figure 4-5.) The fact that the monopolist can work with three control variables calls for three axes to represent all the feasible allocations. We put the first attribute of the good q_1 on the x-axis, and the second attribute q_2 on the y-axis. There would be a third axis projecting out that represents price, but we can omit it, and assume holding price fixed. This does not matter, because we are going to illustrate, by way of contradiction, the sub-optimality of first best allocations, to show that optimal allocations are distorted. Since price is fixed at an arbitrary level, we will establish that the undistorted allocation cannot be optimal for any price. As with the previous section, consumers have utility functions of CRRA class, that are concave with respect to the two attributes of the good. The heterogeneity of the consumers is represented by different slopes of the utility curves. In order to talk about quality distortions away from the first best, it is useful to characterize what the first best allocation looks like in this setting.

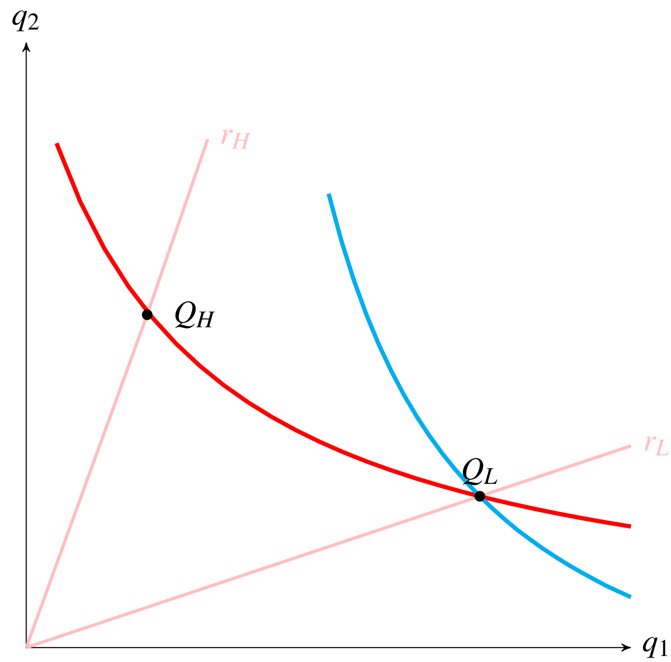


Figure 4-5: Nonlinear pricing with two dimensional goods.

The first best allocation refers to the allocation that would have been optimal had there been no informational asymmetry between monopolist and buyers. Graphically, it is the allocation where iso-profit curves are tangent to the indifference curves, as was the case with the single dimensional goods, or equivalently, where the iso-cost curves are tangent to the indifference curves.[2] More explicitly, for each type, q_1 and q_2 in the first best allocation satisfy

$$\frac{\partial C(q_1,q_2)}{\partial q_1} - \frac{\partial U(q_1,q_2)/\partial q_1}{\partial U(q_1,q_2)/\partial q_2}\frac{\partial C(q_1,q_2)}{\partial q_2} = 0. \tag{4.3}$$

[2]Note that in the previous chapter, we said that the first best allocations are where the utility curves are tangent to the iso profit curves. Here, we are good with using iso-cost curves, since we are assuming holding price fixed.

The utility function U is given by (2.1):

$$\begin{aligned} U_i(q_1,q_2) &= u(b_1^i q_1) + u(b_2^i q_2) - p_i \\ &= \frac{(b_1^i q_1)^{1-\gamma}}{1-\gamma} + \frac{(b_2^i q_2)^{1-\gamma}}{1-\gamma} - p_i \in \{H,L\} \end{aligned} \tag{4.4}$$

where p_i is constant. Suppose also that $C(q_1,q_2) = c_1 q_1 + c_2 q_2$ is linear. Then

$$c_1 - \left(\frac{b_1}{b_2}\right)^{1-\gamma} \left(\frac{q_1}{q_2}\right)^{-\gamma} c_2 = 0$$

$$\frac{q_1^*}{q_2^*} = \left(\frac{c_1}{c_2}\right)^{-1/\gamma} \left(\frac{b_1}{b_2}\right)^{\frac{1-\gamma}{\gamma}} \tag{4.5}$$

Since c_1, c_2, b_1, b_2 are constants, we see that the ratio q_1/q_2 is constant in the first best allocations. Thus, the first best allocations can be represented by a straight ray through the origin. These rays are represented by the pink lines in Figure 4-5.[3]

We further note that in Figure 4-5, as with Figure 4-2, the minimum utility constraint of L type binds, and that the incentive compatibility constraint of H type binds. In the previous section with single dimensional good, this resulted in informational rent for H type and quality deterioration of L type product. In the multidimensional good case, the results are going to be analogous. To see this, we observe that the menu $\{Q_H, Q_L\}$ in Figure 4-5 violates the necessary conditions for the optimal strategy. Note that as with the previous case, moving L's allocation along its indifference curve by an infinitesimal amount does not affect the profits in the first order, since iso-profit curve is tangent to U_L at Q_L. This is represented by a movement from Q_L to Q_L' in Figure 4-6.

As with before, H type consumers, who were previously indifferent between

[3]Figure 4-5, due to dimensional constraint of the graph, assumes the case where the optimal products have the same price. But even if prices are different, the result that L type's product is distorted down persists as long as the H type has shallower indifference curves everywhere, and the preferences are represented by CRRA utility function that is quasi-linear in price.

buying Q_H and Q_L, become strictly better off by buying Q_H to Q'_L. This creates further scope for the monopolist to extract surplus from the H type. Note that the monopolist can do so, through three channels. It can increase the price of the H type product, decrease q_1^H, decrease q_2^H, or use some combination of the three. Then monopolist will decrease q_1^H and q_2^H until H type consumers become indifferent between buying new allocation Q'_H and Q'_L. Therefore we see that profit maximizing screening mechanism for multidimensional products also involves L type's allocation being distorted away from the first best. In fact the this is analogous to the standard result from Armstrong and Rochet (1999) for multi-product nonlinear pricing, which indicates that all 'high-type' consumers (whose incentive compatibility constraint binds) are served efficient quantities, while 'low-type' consumers are served lower quantities than is efficient.

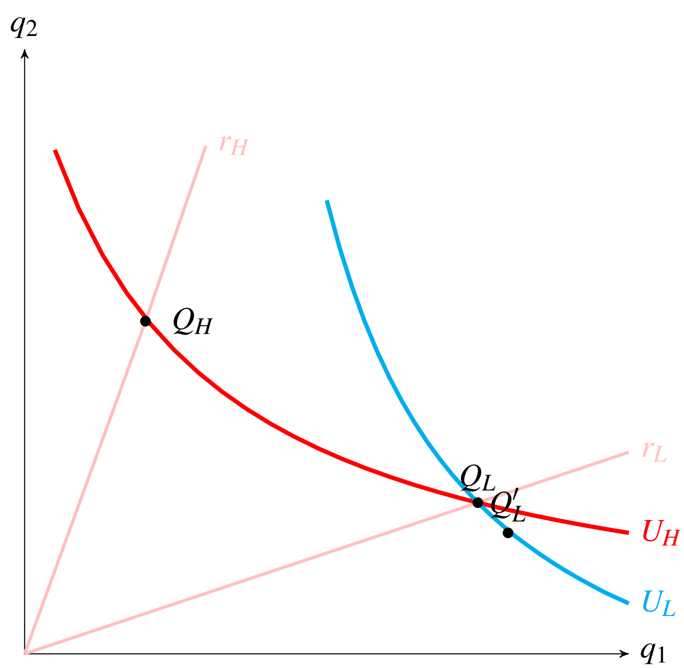


Figure 4-6: Nonlinear pricing with two dimensional goods.

In this chapter, we have seen that in both single dimensional, and two dimensional cases, the optimal product design for a nonlinear pricing monopolist involves undistorted allocation for the 'high type' buyers whose incentive compatibility constraint binds, and distorted allocation for the 'low type' buyers whose incentive compatibility constraint is slack. The first section deals with nonlinear pricing monopolist designing single dimensional good, and demonstrates the sources of informational rent and distortion in quality for the low type. The second section looks at the nonlinear pricing monopolist designing two dimensional good, and shows that the main result from the single dimensional case persists: that is, the high type gets undistorted allocation, while the low type gets distorted allocation.

Chapter 5

Linear Pricing with Multidimensional Good

So far, we have studied the equilibria under two assumptions: (i) that the monopolist can prevent resale, and (ii) that it can observe total amount of goods purchased by a consumer. In fact these two assumptions are precisely the market characteristics that make the monopolist be able to price non-linearly. Airplane seats and hotel rooms, where the buyer has to identify his or her identity at the time of buying, are good examples of markets with nonlinear pricing firms with monopoly power. We now relax these two assumptions and study what the equilibrium looks like, under the assumptions that the monopolist can neither (i) prevent resale, nor (ii) observe total amount purchased by a consumer. The first assumption is rather straightforward–it allows for the existence of secondary market where consumers can resell the goods that they have purchased before at a price other than monopolist price. This rules out scope for bulk discounting for the monopolist, for if there is a bulk discount, one consumer will buy the goods and redistribute them at a secondary market. The second assumption implies that there is a scope for buyers to come back any time to purchase the same product, without being noticed by the monopolist. This rules out bulk surcharges. In a sense, the monopolist tomorrow is a competitor of the monopolist today, since if the monopolist charges more for last few units consumers will choose to buy first few units only and wait until tomorrow to buy first few units again. These assumptions together rule out bulk discounting and charging more for the last few units, and indeed call for linear pricing schedule.

Before talking about screening mechanism with linear pricing schedule, it is useful to note that linear pricing monopolist can only price discriminate between buyers when it can differentiate goods on multidimensional attributes. This is primarily because the monopolists loses control over at least one variable when the market constraints impose linear pricing. Since consumers now solve their utility maximization problems by choosing the number of units to purchase, the monopolist loses one of its control variables that were used to design screening mechanism. To see this, suppose that the monopolist raises the price of a good. With nonlinear pricing, consumers could either take or leave the offer, and cannot buy less of it. However, with linear pricing, consumers can freely choose to buy less of the product when price increases. This is precisely why screening is impossible when the good is single dimensional: consumers, regardless of their types, will always choose to buy a product that gives higher quantity per dollar. With multidimensional goods, however, even if monopolists lose control over price, they still have control over at least two variables, which enables them to sort out two types of buyers with different preferences.

To intuitively see how the differentiated products look like with linear pricing monopolist, I introduce an example of food monopoly on a desert island. The example may sound unrealistic, but it is a good example to see how the theory works, since it captures theoretical exposition in the clearest way. Suppose that the monopolist sells food rations to people on a desert island, and that there are two types of buyers in the island: manual laborers, denoted as the *M* type, and office workers, denoted as the *O* type.

Both manual laborers and office workers need calories and vitamin D's to survive, but laborers need more calories for their work outdoors, and office workers need more vitamin D for their work indoors. Office workers are richer, and have more dollars to spend on rations. Contrary to the examples we have been looking at, the monopolist cannot prevent resale, nor prevent multiple purchases. These assumptions imply that buyers can consume some of both types of rations. They also force the monopolist to price linearly.

The optimal strategy in the food monopoly involves the monopolist designing two rations—one for each type, just as in the airline example. The monopolist would put relatively more vitamin D in the office workers' ration, and distort the manual workers' ration to have too low vitamin D, for the same reason that the airline company distorted the quality of the coach seats. The results are analogous to that of the nonlinear pricing airline company up to this point. The difference is that the monopolist, in this case, will distort the office workers' bundle to have too low vitamin D as well, instead of serving them the first-best allocations, as the airline monopolist did with businessmen. Let us work out the intuition as to why this might be so.

Just as in the airline case, the informational problem that the monopolist faces is the office workers buying manual workers' cheaper rations. In order to prevent this from happening, the monopolist would distort manual workers' rations as before, which partially solves the problem. But there is a new effect here. Suppose that office workers' rations are undistorted, and they get plenty of vitamin D's with their own rations. Then buying a few vitamin D deprived *M* rations wouldn't hurt very much, so it is likely that they would buy some of *M* rations, after they are sufficiently satisfied on vitamin D with their own rations. But if office workers' own rations were already somewhat vitamin D deficient, then buying vitamin deprived M rations would hurt more. So monopolist can raise prices on O's rations and make more profits.

Therefore, if sales are anonymous, and if resale is not preventable, the monopolist optimally distorts both types of rations, and in the same direction. Both office workers and manual laborer's rations get vitamin D deprived.

Let us put this example in the graphical framework. (See Figure 5-1.) Now we have calories as x-axis, and vitamin-D's as y-axis. Office workers' undistorted ray is steeper than manual laborer's undistorted ray, since office workers care more about vitamin D than manual laborers do. Just as in airline case, profit is maximized at the undistorted rays, and if the monopolist could correctly identify the type of buyer at the time of selling, the allocations for both types would lie on the undistorted rays. Also, just as in the airline case, to scare off office workers from buying manual laborers' rations, the manual laborers' ration is distorted down, to be vitamin D deficient. The new effect in the linear pricing case is that office workers' own ration is distorted down to be vitamin D deficient as well. So in the end, both types' rations get distorted, and in the same direction.

Let us unravel this result, and see where the difference in the results between linear pricing and nonlinear pricing are coming from. Recall that the standard incentive compatibility constraint is represented by the indifference curve itself. In other words, if a buyer can choose an allocation that lies above her indifference curve, she has an incentive to deviate. If there's no such allocation, then she is good with what she has. This precisely reflects the setting of the nonlinear pricing monopolist, where buyers cannot buy some of both goods.

However, if buyers can buy $n_H^*/2$[1] goods of their own bundle, and $n_H^*/2$ goods of the other types' bundle, then the utility curve no longer serves as an incentive

[1] n_i^* being the optimal number of units that the i type of buyer ends up buying.

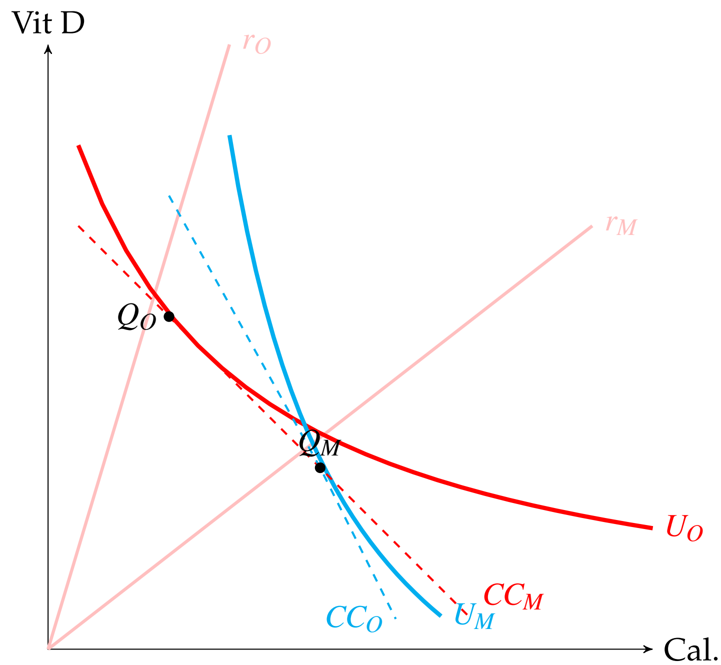


Figure 5-1: Food Monopoly

compatibility constraint. Instead, the correct incentive compatibility constraint to use in this case would be convexification constraint from Rothschild (2014). Convexification constraint is the linear analogue of the standard ones, represented in Figure 5-1 by the dotted lines, CC_O and CC_M. These constraints capture the relaxed market assumptions that buyers can resell goods, and that they can buy some of both types of goods. Let us once again abstract away from the example, and prove the intuition with mathematics.

Suppose that there are two types of buyers H and L in the market with different preferences. Monopolist provides a menu of two goods targeted at each type $\{Q_H, Q_L\}$ where Q_H is the good targeted at the H type, and Q_L is the good targeted at L type. In the nonlinear pricing context, the revelation principle (Myerson, 1979) allows us to focus on direct incentive compatible implementation: the monopolist targets a good for each type, but can only choose incentive compatible targets. In the non-linear pricing case, we can employ the same basic approach, except that the new incentive constraints are tighter, to reflect the fact that the buyers can buy any amount of a given contract, and, moreover, can also choose to 'mix' the contracts. Therefore, we derive indirect utilities that incorporate consumers' utility maximization problem, and consider designing a menu $\{Q_H, Q_L\}$ that is incentive compatible. Let us first solve consumers' utility maximization problem.

Buyers have preferences represented by utility function given by (2.1), which is presented again below:

$$\begin{aligned} U_i(nq_1, nq_2) &= u(b_1 nq_1) + u(b_2 nq_2) - p \times n \\ &= \frac{(b_1^i nq_1)^{1-\gamma}}{1-\gamma} + \frac{(b_2^i nq_2)^{1-\gamma}}{1-\gamma} - n \qquad i \in \{H, L\} \end{aligned} \tag{5.1}$$

U is continuously differentiable, strictly concave, and strictly increasing in q_1 and q_2. Price p is normalized to 1. n measures the number of units that the buyers buy.

Heterogeneity of buyers is represented by two parameters, b_1 and b_2. b_1 measures preference for the first attribute of good, and b_2, likewise, measures preference for the second attribute of good. We assume $\frac{b_1^H}{b_2^H} < \frac{b_1^L}{b_2^L}$. This will ensure that the H type's indirect indifference curves are shallower than L type's indirect indifference curves. Buyers have strictly concave homothetic preferences with constant relative risk aversion. Concavity of preference with respect to q_1 and q_2 ensures diminishing marginal utility, homotheticity ensures that the buyers of the same type demand same proportions of goods given prices, and constant relative risk aversion ensures constant ratio of marginal utility with respect to attributes of the good. The constant ratio of marginal utilities will turn out to be very useful, as it leads to the first best allocations lying on a straight ray from the origin.

Without loss of generality, a given linearly-priced contract can be defined as the vector (q_1, q_2) of quantities per dollar spent on the good. For buyers who are 'forced' to buy one and only one good, the choice they can make is to choose the quantity n of that good that they wish to purchase. Consumer's utility maximization problem then becomes maximizing (5.1) with respect to n. Solving the first order condition, we derive a function of optimal n^* in terms of q_1 and q_2.

$$n^*(q_1, q_2) = \left[(b_1 q_1)^{1-\gamma} + (b_2 q_2)^{1-\gamma}\right]^{\frac{1}{\gamma}} \tag{5.2}$$

Plugging it into the original utility function, we get indirect utility function in terms of q_1 and q_2:

$$V(q_1, q_2) \equiv U(n^* q_1, n^* q_2) = \frac{\gamma}{1-\gamma}\left[(b_1 q_1)^{1-\gamma} + (b_2 q_2)^{1-\gamma}\right]^{\frac{1}{\gamma}} \tag{5.3}$$

Notice that n does not enter into the utility function anymore once buyer has solved utility maximization problem. Since $U(n^* q_1, n^* q_2) = \frac{\gamma}{1-\gamma} n^*(q_1, q_2)$, it is immediately implied that the points on the same indifference curve in (q_1, q_2) space are also points of equal n^*. This allows us to analyze the problem in the (q_1, q_2) plane using indirect utilities.

We note that indifference curves have convex shape on (q_1, q_2) plane, and that changing n does not affect the shape of the indifference curve. We can prove this by proving concavity of V with respect to q_1 and q_2, and that the indifference curves have convex shape follows as an immediate corollary.

Lemma 1. *$V(q_1, q_2)$ is concave with respect to q_1 and q_2.*

Proof. We calculate the Hessian of V given by (5.3), and show that it is negative definite.

$$
\begin{aligned}
HV(q_1,q_2) &\equiv \begin{vmatrix} \frac{\partial^2 V}{\partial q_1^2} & \frac{\partial^2 V}{\partial q_1 \partial q_2} \\ \frac{\partial^2 V}{\partial q_2 \partial q_1} & \frac{\partial^2 V}{\partial q_2^2} \end{vmatrix} \\
&= \frac{\partial^2 V}{\partial q_1^2}\frac{\partial^2 V}{\partial q_2^2} - \frac{\partial^2 V}{\partial q_1 \partial q_2}\frac{\partial^2 V}{\partial q_2 \partial q_1} \\
&= \left[b_1^2 (b_1 q_1)^{-2\gamma} \left((b_1q_1)^{1-\gamma} + (b_2q_2)^{1-\gamma} \right)^{\frac{1}{\gamma}-2} \left(-\gamma (b_1q_1)^{\gamma-1} \left((b_1q_1)^{1-\gamma} + (b_2q_2)^{1-\gamma} \right) + \frac{(1-\gamma)^2}{\gamma} \right) \right] \\
&\times \left[b_2^2 (b_2 q_2)^{-2\gamma} \left((b_1q_1)^{1-\gamma} + (b_2q_2)^{1-\gamma} \right)^{\frac{1}{\gamma}-2} \left(-\gamma (b_2q_2)^{\gamma-1} \left((b_1q_1)^{1-\gamma} + (b_2q_2)^{1-\gamma} \right) + \frac{(1-\gamma)^2}{\gamma} \right) \right] \\
&- \left[(b_1q_1)^{-\gamma} (b_2q_2)^{-\gamma} b_1 b_2 \frac{(1-\gamma)^2}{\gamma} \left((b_1q_1)^{1-\gamma} + (b_2q_2)^{1-\gamma} \right)^{\frac{1}{\gamma}-2} \right]^2 \\
&= \gamma^2 (b_1q_1)^{-\gamma-1} (b_2q_2)^{-\gamma-1} b_1^2 b_2^2 \left((b_1q_1)^{1-\gamma} \right. \\
&\qquad \left. + (b_2q_2)^{1-\gamma} \right)^{\frac{2}{\gamma}-2} + (b_1q_1)^{-2\gamma} (b_2q_2)^{-2\gamma} b_1^2 b_2^2 \frac{(1-\gamma)^4}{\gamma^2} \left((b_1q_1)^{1-\gamma} + (b_2q_2)^{1-\gamma} \right)^{\frac{2}{\gamma}-4} \qquad 2 \\
&\qquad - \gamma (b_1q_1)^{-\gamma-1} (b_2q_2)^{-2\gamma} b_1^2 b_2^2 \left((b_1q_1)^{1-\gamma} + (b_2q_2)^{1-\gamma} \right)^{\frac{2}{\gamma}-3} \\
&\qquad - \gamma (b_2q_2)^{-\gamma-1} (b_1q_1)^{-2\gamma} b_1^2 b_2^2 \left((b_1q_1)^{1-\gamma} + (b_2q_2)^{1-\gamma} \right)^{\frac{2}{\gamma}-3} \\
&\qquad - (b_1q_1)^{-2\gamma} (b_2q_2)^{-2\gamma} b_1^2 b_2^2 \frac{(1-\gamma)^4}{\gamma^2} \left((b_1q_1)^{1-\gamma} + (b_2q_2)^{1-\gamma} \right)^{\frac{2}{\gamma}-4} \\
&= \gamma^2 b_1^2 b_2^2 \left((b_1q_1)^{1-\gamma} + (b_2q_2)^{1-\gamma} \right)^{\frac{2}{\gamma}-3} (b_1q_1)^{-\gamma-1} (b_2q_2)^{-\gamma-1} \\
&\qquad\qquad \times \left[\left((b_1q_1)^{1-\gamma} + (b_2q_2)^{1-\gamma} \right) - \frac{1}{\gamma} \left((b_1q_1)^{1-\gamma} + (b_2q_2)^{1-\gamma} \right) \right] \\
&= \gamma (1-\gamma) b_1^2 b_2^2 \left((b_1q_1)^{1-\gamma} + (b_2q_2)^{1-\gamma} \right)^{\frac{2}{\gamma}-2} (b_1q_1)^{-\gamma-1} (b_2q_2)^{-\gamma-1} < 0
\end{aligned}
$$

Last term is negative since $\gamma > 1$ by assumption. We also verify that $\frac{\partial^2 V}{\partial q_1^2}$ is negative.

Since

$$
\frac{\partial^2 V}{\partial q_1^2} = b_1^2 (b_1 q_1)^{-2\gamma} \left((b_1q_1)^{1-\gamma} + (b_2q_2)^{1-\gamma} \right)^{\frac{1}{\gamma}-2} \left(-\gamma (b_1q_1)^{\gamma-1} \left((b_1q_1)^{1-\gamma} + (b_2q_2)^{1-\gamma} \right) + \frac{(1-\gamma)^2}{\gamma} \right),
$$

we only need to verify that

$$-\gamma(b_1q_1)^{\gamma-1}\left((b_1q_1)^{1-\gamma}+(b_2q_2)^{1-\gamma}\right)+\frac{(1-\gamma)^2}{\gamma}<0$$

Rearranging,

$$-\gamma(b_1q_1)^{\gamma-1}\left((b_1q_1)^{1-\gamma}+(b_2q_2)^{1-\gamma}\right)+\frac{(1-\gamma)^2}{\gamma} \quad =-\gamma\left[1+\left(\frac{b_1q_1}{b_2q_2}\right)^{\gamma-1}-\left(\frac{1-\gamma}{\gamma}\right)^2\right]<0$$

where the inequality follows from the fact that $0<\left(\frac{1-\gamma}{\gamma}\right)<1$ since $\gamma>1$ by assumption. □

Before stating the profit maximization problem of the monopolist, we note that in the linear pricing case, the difference in assumptions from the nonlinear pricing case translates into a different set of market constraints. In particular, the canonical incentive compatibility constraints IC_H and IC_L no longer serve their roles as market constraints in the linear pricing case, since the ability of the buyers to freely take linear combinations of the contracts undermines their functional forms. To see what the new consumer utility maximization problem could do to the canonical incentive compatibility constraint, and to recognize the need for a new incentive compatibility constraint, we look at Figure 5-2. Let us show, by way of contradiction, the inaptness of canonical incentive constraints heuristically. Suppose that the allocations Q_H and Q_L are profit maximizing optimal allocation designed by the monopolist. We see that Q_H and Q_L clearly satisfy the canonical incentive compatibility constraints, as for each type, the allocation of the other type lies on or to the lower left to her indifference curve. However, this menu of allocations is not feasible with linear pricing, since a buyer of H type could buy bundles such as Q'_H, buy spending $\frac{n_H^*}{2}$ on Q_H and $\frac{n_H^*}{2}$ on Q_L.[2] Such an allocation is feasible, since

[2] n_H^* is the same for H at Q_H and Q_L.

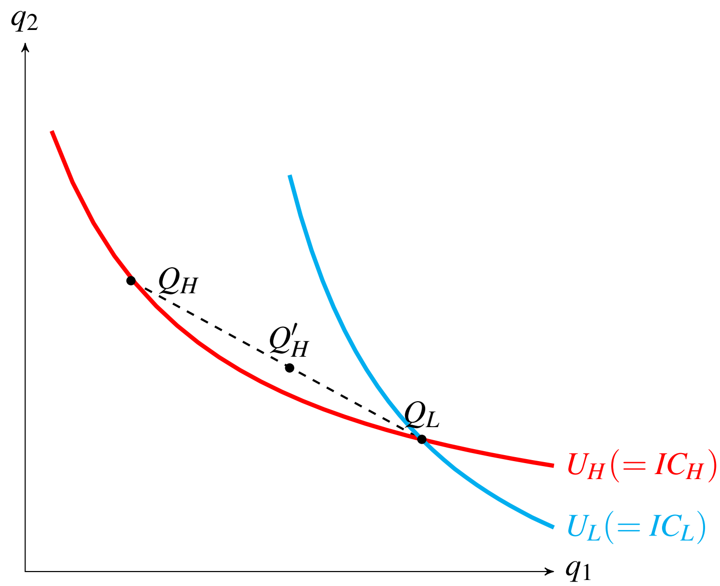


Figure 5-2: Linear pricing undermines optimum derived with canonical incentive compatibility constraints.

the set of feasible allocation is represented by the line segment $[Q_H, Q_L]$. In fact, Q'_H would give her higher utility than Q_H, so she clearly has incentive to deviate from her targeted product. We can lay out the mathematics more explicitly using (5.3). Suppose $V_H(q_1^H, q_2^H) = V_H(q_1^L, q_2^L)$, where $Q_H = (q_1^H, q_2^H), Q_L = (q_1^L, q_2^L)$, and $Q'_H = \left(\frac{q_1^H+q_1^L}{2}, \frac{q_2^H+q_2^L}{2}\right)$. Then we claim that $V_H(q_1^H, q_2^H) \leq V_H\left(\frac{q_1^H+q_1^L}{2}, \frac{q_2^H+q_2^L}{2}\right)$ with equality, if and only if $q_1^H = q_1^L$ and $q_2^H = q_2^L$.

$$
\begin{aligned}
V_H\left(\frac{q_1^H+q_1^L}{2}, \frac{q_2^H+q_2^L}{2}\right) &= \frac{\gamma}{1-\gamma}\left[\left(b_1^H \frac{q_1^H+q_1^L}{2}\right)^{1-\gamma} + \left(b_2^H \frac{q_2^H+q_2^L}{2}\right)^{1-\gamma}\right]^{1/\gamma} \\
&\geq \frac{1}{2}\frac{\gamma}{1-\gamma}\left[(b_1^H q_1^H)^{1-\gamma} + (b_2^H q_2^H)^{1-\gamma})\right]^{1/\gamma} + \frac{1}{2}\frac{\gamma}{1-\gamma}\left[(b_1^H q_1^L)^{1-\gamma} + (b_2^H q_2^L)^{1-\gamma})\right]^{1/\gamma} \\
&= \frac{\gamma}{1-\gamma}\left[(b_1^H q_1^H)^{1-\gamma} + (b_2^H q_2^H)^{1-\gamma})\right]^{1/\gamma} \\
&= V_H(q_1^H, q_2^H)
\end{aligned}
$$

where in the third last inequality the concavity of $V(\cdot)$ is used, and in the second last equality, the fact that $V_H(q_1^H, q_2^H) = V_H(q_1^L, q_2^L)$ is used. Since buyers can take linear combinations of the available products Q_H and Q_L to reach allocations like Q'_H, allocations such as Q_H and Q_L are not incentive compatible in the linear pricing context. In order to solve the problem such that no type has incentive to deviate, the monopolist would need a different set of incentive compatibility constraints that take these consumer decisions into account. Intuitively, we need to ensure that neither type has an incentive to move 'towards' the other type's allocation.

The correct incentive compatibility constraints to use in this case is the convexification constraint from Rothschild (2014), that addresses this problem directly by including every possible linear combinations of the allocations in the incentive compatible sets. Figure 5-3 shows a feasible allocation Q_H and Q_L, under convexification constraints, which are represented by dashed lines CC_H and CC_L.

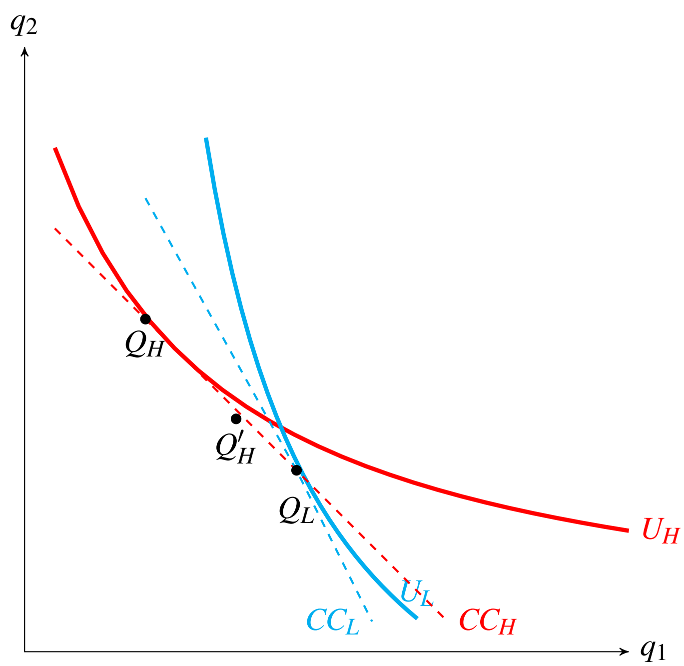


Figure 5-3: Convexification constraint

Suppose that an H type consumer, as in the previous paragraph, tries to take a linear combination of the two allocations Q_H and Q_L, in the hope that she would arrive at an allocation Q'_H that gives her higher utility than Q_H that is targeted at her. We see that whichever combination she takes, it is going to lie on or to the lower left of the convexification constraint CC_H. In fact, if the allocations Q_H and Q_L were solutions to the profit maximization problem, it would be implemented without any type deviating from the product targeted at her, which would result in the monopolist successfully engaging in second degree price discrimination. We thus have the following lemma:

Lemma 2. *A direct allocation Q_H, Q_L is incentive compatible in linear pricing if and only if (CC_H) and (CC_L) are satisfied, where*

$$b_1^H u_H'(b_1^H q_1^H)(q_1^L - q_1^H) + b_2^H u_H'(b_2^H q_2^H)(q_2^L - q_2^H) \leq 0 \qquad (CC_H)$$

$$b_1^L u_L'(b_1^L q_1^L)(q_1^H - q_1^L) + b_2^L u_L'(b_2^L q_2^L)(q_2^H - q_2^L) \leq 0 \qquad (CC_L)$$

We prove this lemma to establish that this is the correct incentive compatibility constraint to use, and then consider profit maximization problem using this constraint. To prove this lemma, we consider a movement along the line segment between Q^H and Q^L and check whether it is incentive compatible or not. To formalize, fix Q^L on a (q_1, q_2) plane, and consider any line through Q^L. Then fix any $Q^H \neq Q^L$ on that line and let any point on the $[Q^H, Q^L]$ segment be noted by $Q^H(\varepsilon)$ where $0 \leq \varepsilon \leq 1$ so that $Q^H(0) = Q^H$ and $Q^H(1) = Q^L$. Define $\hat{V}(\varepsilon) = V(q(\varepsilon))$. We first prove a lemma that $\hat{V}(\varepsilon)$ is strictly concave.

Lemma 3. $\hat{V}(\varepsilon)$ *is strictly concave.*

Proof. We need to show that $\frac{\partial^2 \hat{V}}{\partial \varepsilon^2} < 0$ to prove concavity of $\hat{V}$.

$$\frac{\partial \hat{V}}{\partial \varepsilon} = \left[(b_1 q_1(\varepsilon))^{1-\gamma} + (b_2 q_2(\varepsilon))^{1-\gamma}\right]^{\frac{1}{\gamma}-1} \left[(b_1 q_1(\varepsilon))^{-\gamma} b_1 \frac{\partial q_1(\varepsilon)}{\partial \varepsilon} + (b_2 q_2(\varepsilon))^{-\gamma} b_2 \frac{\partial q_2(\varepsilon)}{\partial \varepsilon}\right]$$

$$\frac{\partial^2 \hat{V}}{\partial \varepsilon^2} = \left(\frac{1}{\gamma} - 1\right) \left[b_1 q_1(\varepsilon))^{1-\gamma} + (b_2 q_2(\varepsilon))^{1-\gamma}\right]^{\frac{1}{\gamma}-2} \left[(b_1 q_1(\varepsilon))^{-\gamma} b_1 \frac{\partial q_1(\varepsilon)}{\partial \varepsilon} + (b_2 q_2(\varepsilon))^{-\gamma} b_2 \frac{\partial q_2(\varepsilon)}{\partial \varepsilon}\right]^2$$
$$+ \left[b_1 q_1(\varepsilon))^{1-\gamma} + (b_2 q_2(\varepsilon))^{1-\gamma}\right]^{\frac{1}{\gamma}-1} (-\gamma) \left[(b_1 q_1(\varepsilon))^{-\gamma-1} \left(b_1 \frac{\partial q_1(\varepsilon)}{\partial \varepsilon}\right)^2 + (b_2 q_2(\varepsilon))^{-\gamma-1} \left(b_2 \frac{\partial q_2(\varepsilon)}{\partial \varepsilon}\right)^2\right]$$

Noting that $\frac{\partial q_1(\varepsilon)}{\partial \varepsilon}$ and $\frac{\partial q_2(\varepsilon)}{\partial \varepsilon}$ are constants, we see that the second term is negative, and the first term is negative as long as $\frac{1}{\gamma} - 1 < 0$, or $\gamma > 1$. [3] Thus $\frac{\partial^2 \hat{V}}{\partial \varepsilon^2} < 0$, and since the choice of ε was arbitrary, $\hat{V}(\varepsilon)$ is strictly concave everywhere. □

We note that the sign of LHS of CC_H is equal to the sign of $\frac{\partial \hat{V}_H}{\partial \varepsilon}$. Explicitly,

$$\begin{aligned}\frac{\partial \hat{V}_H}{\partial \varepsilon} &= \left[(b_1^H q_1^H(\varepsilon))^{1-\gamma} + (b_2^H q_2^H(\varepsilon))^{1-\gamma}\right]^{\frac{1}{\gamma}-1} \left[(b_1^H q_1^H(\varepsilon))^{-\gamma} b_1^H \frac{\partial q_1^H(\varepsilon)}{\partial \varepsilon} + (b_2^H q_2^H(\varepsilon))^{-\gamma} b_2^H \frac{\partial q_2^H(\varepsilon)}{\partial \varepsilon}\right] \\ &= \left[(b_1^H q_1^H(\varepsilon))^{1-\gamma} + (b_2^H q_2^H(\varepsilon))^{1-\gamma}\right]^{\frac{1}{\gamma}-1} \left[b_1^H (b_1^H q_1^H)^{-\gamma}(q_1^L - q_1^H) + b_2(b_2 q_2)^{-\gamma}(q_2^L - q_2^H)\right]\end{aligned}$$

and

$$CC_H : b_1^H u_H'(b_1^H q_1^H)(q_1^L - q_1^H) + b_2^H u_H'(b_2^H q_2^H)(q_2^L - q_2^H) \leq 0$$

Therefore, to claim that incentive compatibility is satisfied if and only if CC_H holds, it suffices to show that incentive compatibility is satisfied if and only if $\hat{V}'(0) \leq 0$.

Lemma 4. *H is optimal on the line segment ($\hat{V}'(0) \leq 0$), if and only if CC_H is satisfied.*

Proof. If $\hat{V}'(0) > 0$, then for a small enough ε, H type prefers $(1-\varepsilon)q^H + \varepsilon q^L$ to q^H, so incentive compatibility fails. If $\hat{V}'(0) < 0$, then by quasi concavity of V, anything on the line segment is worse than q_H. Thus incentive compatibility is satisfied. If $\hat{V}'(0) = 0$, then since the set of $\{q | V(q) > V(q^H)\}$ is strictly convex, any point on the line segment lies outside of the set, so H type has no incentive to deviate. Thus incentive compatibility is satisfied. □

Now that we are equipped with the correct incentive compatibility constraint to use in the linear pricing case, we can use this to solve profit maximization problem of a linear pricing monopolist. Profit maximization problem that the monopolist solves can be stated more formally as follows:

[3] In fact, the same result would hold for $\gamma \leq 1$. In other words, concavity of $\hat{V}(\varepsilon)$ doesn't matter for the convexification constraints to be valid. The necessary condition is that $\hat{V}(\varepsilon)$ is single peaked, which follows from quasi-concavity.

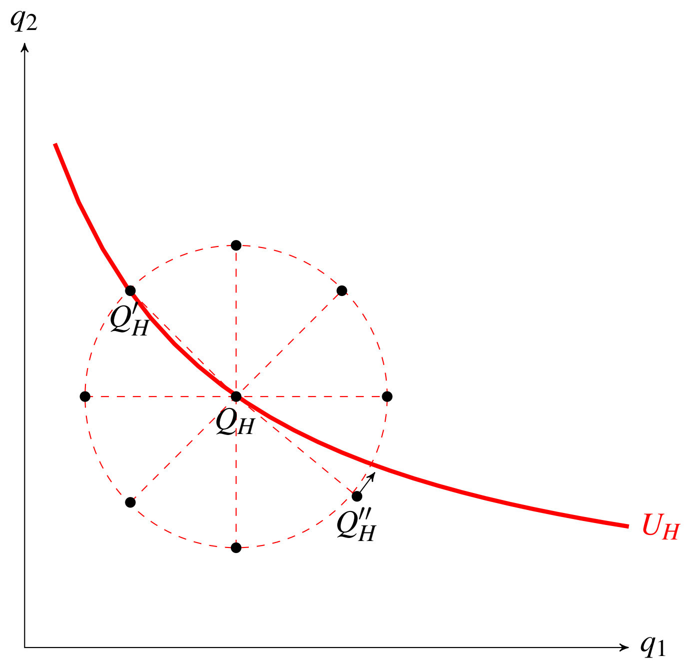


Figure 5-4: Illustration of Lemma 4.

$$\max_{Q_i,p_i} \Pi(n_i, Q_i, p_i) = \lambda\left[n_H(Q_H)\times p_H - C(n_H Q_H)\right] + (1-\lambda)\left[n_L(Q_L)\times p_L - C(n_L Q_L)\right], i\in\{H,L\}$$

$$\text{s.t. } b_1^H u'_H(b_1^H q_1^H)(q_1^L - q_1^H) + b_2^H u'_H(b_2^H q_2^H)(q_2^L - q_2^H) \le 0 \qquad (CC_H)$$

$$b_1^L u'_L(b_1^L q_1^L)(q_1^H - q_1^L) + b_2^L u'_L(b_2^L q_2^L)(q_2^H - q_2^L) \le 0 \qquad (CC_L)$$

$$u_i(q_i) \ge \bar{U}_i, \qquad i\in\{H,L\} \qquad (MU_i)$$

$\lambda \in (0,1)$ is the proportion of H type consumers in the population, known to monopolist. The minimum utility constraints MU_H and MU_L ensure full participation of consumers. Contrary to the nonlinear pricing problem, incentive compatibility constraints are represented by the convexification constraints CC_H and CC_L, which prevent buyers from *convexifying*, or taking linear combinations of the goods offered in the market.

The monopolist faces a linear cost function, which is weakly convex. Weak convexity of the cost function, together with strict concavity of the utility function, ensures unique solution to the profit maximizing problem in the absence of informational asymmetry. We choose functional forms of utility function as (5.1) and linear cost function as below:

$$C(nq_1, nq_2) = c_1 nq_1 + c_2 nq_2 \tag{5.4}$$

C is weakly convex, and increasing in q_1, and q_2.

We first identify the first-best allocations, and analyze quality distortions involved in the optimal allocations, in the presence of informational asymmetry. The first best allocations lie on a straight ray through the origin, and is defined by (4.5), with the utility function and the cost function defined by (5.1) and (5.4). The derivation is repeated nevertheless, for convenience. Suppose that there is one type of consumer in the market, and that there is no informational asymmetry between consumers and the monopolist. Then the firm solves the following profit maximization problem

$$\max_{q_1,q_2} n^* \left[1 - (c_1 q_1 + c_2 q_2)\right]$$

where price is normalized to 1, and C is defined as above. The solution to this problem is equivalent to solution to the following cost minimization problem

$$\min_{q_1,q_2} n^* (c_1 q_1 + c_2 q_2)$$

Since n^* is constant along an indifference curve, we can treat it as constant and take first order condition of C with respect to q_1 and q_2 to get

$$\frac{\partial C(q_1,q_2)}{\partial q_1} - \frac{\partial V/\partial q_1}{\partial V/\partial q_2} \frac{\partial C(q_1,q_2)}{\partial q_2} = 0$$

which gives

$$\frac{q_1^*}{q_2^*} = \left(\frac{c_1}{c_2}\right)^{-1/\gamma} \left(\frac{b_1}{b_2}\right)^{\frac{1-\gamma}{\gamma}}$$

This has two important implications: (1) A ray on (q_1, q_2) plane that goes through the origin with slope r^* characterizes the first best allocations; (2) given a utility level, profit is maximized at the undistorted ray. I refer to the ray of first best allocations "undistorted ray" throughout. The second point seems redundant in the view that r^* is what we got from solving the first order conditions, but it could be used to establish the following result.

Lemma 5. *Points on the undistorted ray solve the first best problem of maximizing profits subject to a given utility level.*

Proof. That the allocations on the rays are profit maximization points to each utility level is established, so we only need to show that it is unique. To establish uniqueness, we note that the indirect indifference curve is convex, and fixing n, iso-costs are lines (with linear costs). And n is constant along each indifference curve. □

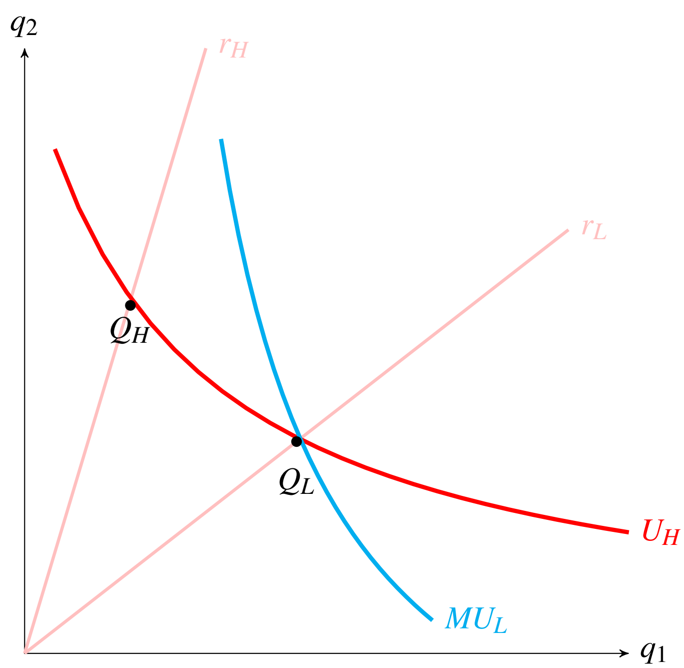


Figure 5-5: Illustration of undistorted rays on which the first best allocations Q_H and Q_L lie.

We have just looked at characteristics of the first best undistorted allocations for a linear pricing monopolist. When there is no informational asymmetry, profit maximizing strategy involves implementing the first best allocations for each type, as noted in the previous chapters. We now characterize an optimal allocation determined by the monopolist in the presence of informational asymmetry. We claim that at an optimal allocation, it is necessary that both H type's and L type's allocations are distorted down when CC_H binds, and that both allocations are distorted up when CC_L binds.

Theorem 1. *1 Let* (q_1^H, q_2^H) *and* (q_1^L, q_2^L) *be the constrained efficient allocation of non-exclusive linear pricing monopolist. Then this allocation falls into one of the three categories below:*

(i)	*CC_H binds and CC_L slack.*	$\frac{q_2^H}{q_1^H} < r_H$ *and* $\frac{q_2^L}{q_1^L} < r_L$
(ii)	*CC_L binds and CC_H slack.*	$\frac{q_2^H}{q_1^H} > r_H$ *and* $\frac{q_2^L}{q_1^L} > r_L$
(iii)	*Neither CC_H nor CC_L bind.*	$\frac{q_2^H}{q_1^H} = r_H$ *and* $\frac{q_2^L}{q_1^L} = r_L$

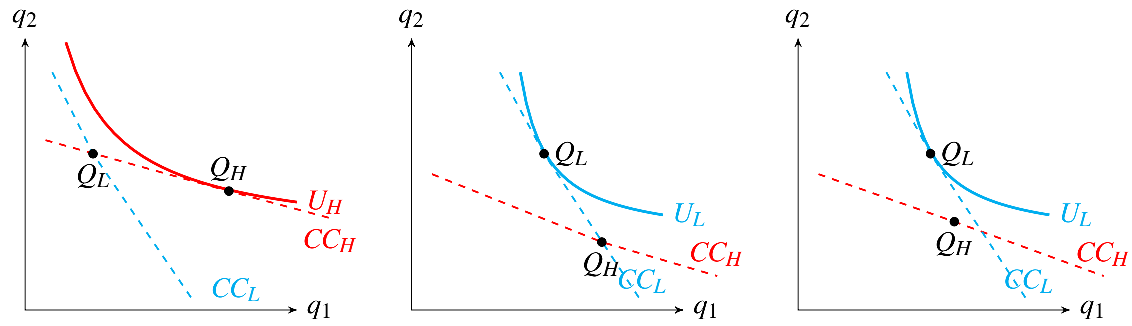
Notice that (i) and (ii) consider cases where either CC_H or CC_L binds, whereas (iii) considers cases where neither CC_H nor CC_L binds. In the light of the fact that the primary incentive for a monopolist to distort allocation comes from the binding incentive compatibility constraints of the buyers, it is natural to see that the first best allocations get implemented when neither of the incentive compatibility constraints bind. Since both CC_H and CC_L are slack in this case, moving allocations away from the first best does nothing but hurting profits. The interesting cases, where the distortions are involved, are the cases (i) and (ii) where incentive compatibility constraint of each type binds.

Since (i) and (ii) are qualitatively analogous, I only prove (i) here. (ii) could be proved by a symmetric argument. (i) says that if CC_H binds and CC_L is slack, then an optimal contracts will both be distorted downwards from the first best. i.e. $\frac{q_2^H}{q_1^H} < r_H$ and $\frac{q_2^L}{q_1^L} < r_L$. We establish this by the following four steps. **Step 1**: Show that Q_L is below and to the right of Q_H. **Step 2**: Show that Q_H is below the undistorted ray, that is, $\frac{q_2^H}{q_1^H} < r_H$. **Step 3**: Show that Q_L is strictly below the undistorted ray r_L. **Step 4**: Show that CC_L does not bind at $\frac{q_2^L}{q_1^L} < r_L$.

Step 1 (Lemma 6): Let us first establish Step 1, that Q_L is below and to the right of Q_H. By the way of contradiction, suppose that an allocation Q_L is above and to the left of Q_H; we will show that such allocation is not feasible.[4] Because the indifference curve for the L type is steeper than that of the H type, and indifference curves are convex, when Q_L is above and to the left of Q_H, it must either be the case that (i) CC_H binds and CC_L is slack, (ii) CC_L binds and CC_H is slack, or (iii) both CC_H and CC_L are slack. We consider each case one by one. If CC_H binds and CC_L is slack, then Q_L is not incentive compatible for the L type buyers since Q_H lies inevitably outside of incentive compatibility zone of the L type, due to the slope of CC_L being steeper than that of CC_H. (See Figure 5-6 (a).) Similarly, if CC_L binds and CC_H is slack, then Q_H is not incentive compatible for the H type buyers, since Q_L lies outside of incentive compatibility zone of the H type. Thus the H type has incentive to deviate to Q_L from Q_H. Let us next consider the last case where neither of the convexification constraints bind. Then the following two things must both be true: Q_H lies below and to the left of CC_L, and Q_L lies below and to the left of CC_H. But it is immediately noted that they cannot both be true, since CC_L is always steeper than CC_H when Q_L lies above and to the left of Q_H. Figure 5-6 (c) illustrates that when Q_H lies below and to the left of CC_L, it cannot also be the case that Q_L lies below and to the left of CC_H. We thus have established that all of the three possible cases that might arise when Q_L lies above and to the left of Q_H are not feasible. There is a contradiction, thus it must be the case that Q_L lies below and to the right of Q_H at all times.

From Step 1, we established that feasible allocation involves Q_L being lower

[4]Note that the result of this step holds generally, and that we do not assume anything about CC_H or CC_L binding in this step, contrary to other following steps, where we assume that one of the incentive compatibility constraint binds.



(a) CC_H binds, CC_L slack. (b) CC_L binds, CC_H slack. (c) Both CC_H and CC_L slack.

Figure 5-6: Illustration of Step 1 Proof.

right to Q_H. We now assume throughout that CC_H binds, and show that both types' allocations are distorted at an optimal allocation. To achieve the results, we consider small movement in H type allocation and/or L type allocation and see what it does to the Lagrangian for the monopolist's problem. The monopolist has to consider five factors when making such movement: change in profit, change in convexification constraint, and whether the minimum utilities of the two types are still satisfied. Thus the Lagrangian is defined as

$$\mathcal{L} = \lambda\Pi_H(Q_H) + (1-\lambda)\Pi_L(Q_L) + \mu\Lambda(Q_H, Q_L) + \zeta\Gamma(Q_L, Q_H) + \phi(U_H - \bar{U}_H) + \tau(U_L - \bar{U}_L)$$

where $\Lambda(Q_H, Q_L)$ is CC_H, $\Gamma(Q_L, Q_H)$ is CC_L, λ is the fraction of the H type in the population, and $\mu \geq 0, \zeta \geq 0, \phi \geq 0, \tau \geq 0$ are Lagrange multipliers associated with the profit function, CC_H, CC_L, MU_H, and MU_L respectively.

Step 2 (Lemma 7): We next establish Step 2, that if CC_H binds at an optimum allocation, Q_H must be below the undistorted ray, or $\frac{q_2^H}{q_1^H} < r_H$. To proceed, we assume by the way of contradiction that $\frac{q_2^H}{q_1^H} \geq r_H$ and consider the effect of a small movement of (q_1^H, q_2^H) along the indifference curve on Lagrangian. If $\frac{q_2^H}{q_1^H} \geq r_H$ does not bind, a small movement along CC_H towards L's allocation weakly improves profit. That is, if $\frac{q_2^H}{q_1^H} > r_H$, profit is improved as H type's allocation moves towards the first best allocation lying on r_H, and if $\frac{q_2^H}{q_1^H} = r_H$, profit is not affected to first order by the movement. Thus, $\lambda\nabla\Pi_H(Q_H) \geq 0$. Next we see how the movement makes CC_H slack, or $\mu\nabla\Lambda(Q_H, Q_L) > 0$. Consider a small movement of Q_H along the indifference curve. Then minimum utility constraint is not affected in the first order because the movement is tangent to the constraint, but CC_H becomes slack as the slope becomes shallower and L type's allocation becomes strictly below to the left of CC_H. (See Figure 5-7.) If CC_L does not bind, then a small movement in H does not affect CC_L. If CC_L does bind, then a small movement in H does not affect CC_L to first order, because the movement is tangent to the indifference curve and thus along CC_L. Therefore, the value of the Lagrangian strictly increases with this movement (all 5 terms nonnegative, and CC_H strictly positive), and thus $\frac{q_2^H}{q_1^H} \geq r_H$ is not optimal. Thus when CC_H binds, the optimal H type allocation is distorted down to the right of the undistorted ray.

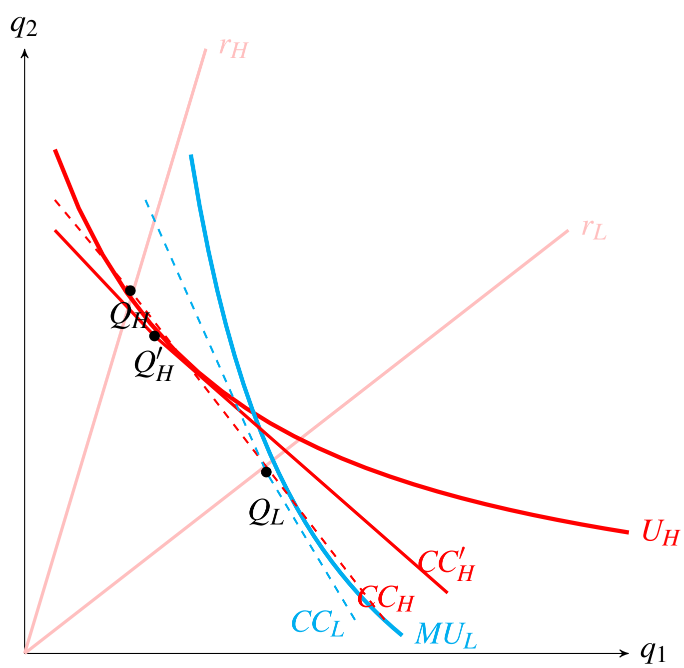


Figure 5-7: Illustration of Step 2 Proof.

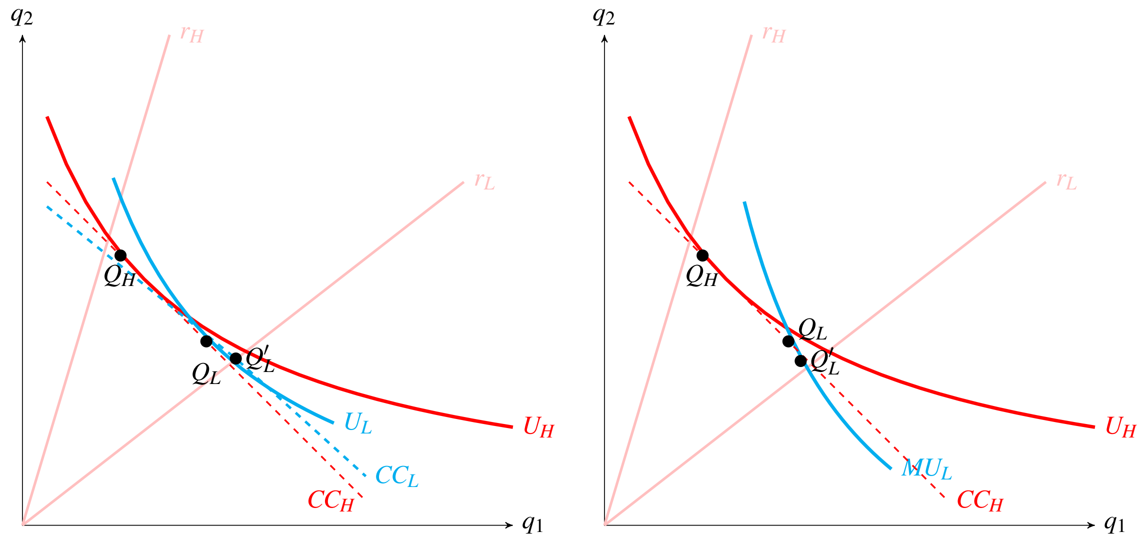
Now that we have established that H type's allocation is always distorted when CC_H binds, we now look at L type's allocation and see whether it must get distorted, and if it does, where it lies with respect to the first best allocation. To show this, we proceed by looking at allocations $\frac{q_2^L}{q_1^L} \geq r_L$ and $\frac{q_2^L}{q_1^L} < r_L$, one at a time. Step 3 establishes that Q_L is strictly below r_L at an optimal allocation. In other words, an optimal Q_L satisfies $\frac{q_2^L}{q_1^L} < r_L$.

Step 3 (Lemma 8 and Corollary 1): To show that $\frac{q_2^L}{q_1^L} < r_L$, we assume $\frac{q_2^L}{q_1^L} \geq r_L$ and show that it is inconsistent with an optimum. To establish this, we first show that CC_L cannot bind at some allocation, and then show that when CC_L does not bind at $\frac{q_2^L}{q_1^L} \geq r_L$, the monopolist can always do better by making small movement of the L type's allocation towards r_L.

Let us first show that an allocation where CC_L binds and $\frac{q_2^L}{q_1^L} \geq r_L$ is not feasible. If both CC_H and CC_L bind, it means that they share the same slope and that convexification constraints are represented by a single line on which Q_H and Q_L lie. So the question boils down to whether CC_H and CC_L sharing the same slope is feasible for Q_H and Q_L lying 'inside' the undistorted rays. It is clearly not feasible: homothetic preferences indicate that the slopes of CC_H and CC_L are constant along the undistorted rays. Furthermore, since the indifference curve of the H type is shallower than that of L type, the slope of CC_H is shallower than the slope of CC_L on the undistorted rays. Note as we move Q_H and Q_L 'inwards,' the slope of CC_H becomes shallower and the slope of CC_L becomes steeper. They can never be equal by construction of the utilities of the two types. Therefore when $\frac{q_2^L}{q_1^L} \geq r_L$, feasible allocation involves CC_L not binding.

We next show that an L type allocation where $\frac{q_2^L}{q_1^L} \geq r_L$ and CC_L does not bind, is suboptimal. From Figure 5-8(b), let us consider a small movement of Q_L along L type's indifference curve. Profit is weakly improved as $\nabla\Pi_L(Q_L) > 0$ when $\frac{q_2^L}{q_1^L} > r_L$ and $\nabla\Pi_L(Q_L) = 0$ when $\frac{q_2^L}{q_1^L} = r_L$.[5] The minimum utility constraint for the L type does not get affected in the first order. Since Q_L' is below to the left of the convexification constraint of the H type, CC_H becomes slack, and it follows that this

[5]This means that the profit is increasing($\nabla\Pi_L(Q_L) > 0$), or locally constant ($\nabla\Pi_L(Q_L) = 0$) with respect to this movement.



(a) Step 3 (i): CC_L cannot bind

(b) Step 3 (ii): CC_L does not bind

Figure 5-8: Illustration of Step 3 Proof.

movement increases the Lagrangian, thus such allocation cannot be an optimum. This completes the proof of Step 3.

So far, we have concluded everything about the locations of optimal Q_H and Q_L on (q_1, q_2) plane: Q_H lies below and to the right of r_H, and Q_L lies below and to the right of r_L. That is, both types' allocations get distorted to the lower right when CC_H binds. We finally say something more about incentive compatibility of the L type on this allocation and conclude our proof. Step 4 in the next paragraph shows that if CC_H binds and $\frac{q_2^H}{q_1^H} < r_H$, then CC_L cannot bind at allocations where $\frac{q_2^L}{q_1^L} < r_L$. In other words, CC_L cannot bind at such allocations.

Step 4 (Lemma 9): We next show that CC_L must be slack at any optimum in which CC_H binds — and hence, by steps 1-3 above, $\frac{q_2^H}{q_1^H} < r_H$ and $\frac{q_2^L}{q_1^L} < r_L$. We suppose that CC_L does bind, by way of contradiction. From Figure 5-9, suppose the monopolist makes a small movement of L type's allocation towards r_L. Then it clearly improves profit as the allocation becomes closer to the undistorted ray, or the first best allocation. Next we look at whether the convexification constraints are affected by this movement. CC_H is not affected in the first order, as the direc-

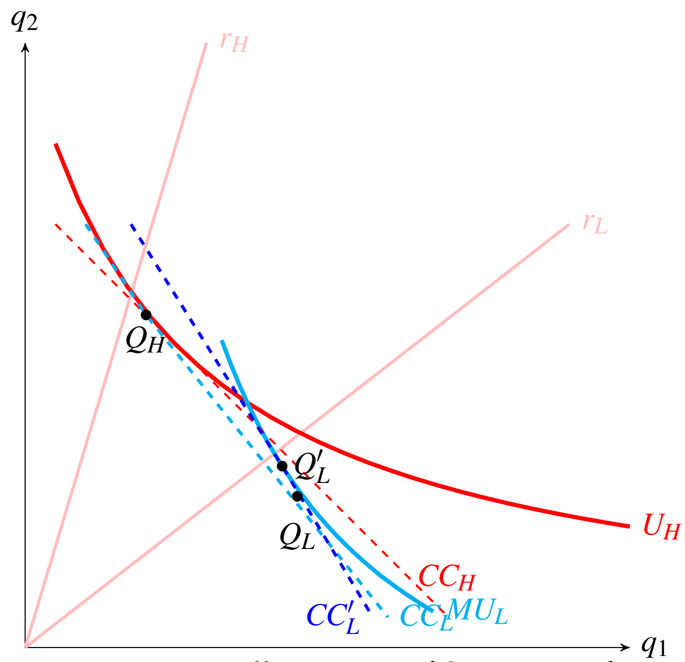


Figure 5-9: Illustration of Step 4 Proof.

tion of the movement is tangent to CC_H, which has the same slope as CC_L when both convexification constraints bind. CC_L, however, becomes slack as L's allocation moves towards r_L. Q_H, which used to lie on CC_L, now lies strictly below and to the left of CC_L due to the small movement. Therefore this increases the Lagrangian, and such allocation cannot be optimal. Thus it must be that CC_L does not bind at such allocations.

We have thus fully established that in the optimal product design for a linear pricing monopolist, allocations get distorted for both types in the same direction when either of the incentive compatibility constraint binds.

Chapter 6

Conclusion

The main implication of this paper is that, when the monopolist facing different types of buyers in the market can neither prevent resale nor observe the total amount of goods purchased by each buyer, the optimal product design involves the monopolist distorting the products for both types of buyers in the same direction. While the literatures on the canonical nonlinear pricing monopolist suggest that efficient screening involves providing undistorted first-best allocation to one and only one type of buyers who have the highest willingness to pay for a good given quality when single crossing property holds, the ability of the buyers to choose the number of units to buy, in the linear pricing case, further encourages the monopolist to distort the quality of the highest type's good to ensure that the highest type does not have any incentive to deviate from her own product. The result suggests that inefficient distortions are more pervasive with linear pricing, when buyers can freely buy some of both goods, than otherwise.

There are two main restrictions in the analysis. First, most of the proofs are executed by imposing specific functional forms on the utility function and the cost function. As for the utility function, although the nice algebraic properties of the problem mostly come from the properties of CRRA utility function, we note that the main result persists as long as the preferences of the buyers are homothetic. The more restrictive assumption is the linearity of the cost function, which is assumed mainly because of the need for additivity when dealing with multiple types of buyers. The simplicity of the exposition makes the application a non-trivial problem, but it nevertheless renders a useful insight as to why the linear pricing monopolist would have an incentive to deliberately degrade the quality for both types of buyers in the market when it cannot observe the types of buyers at the time of selling.

Another big restriction implicit in the analysis of this paper is the two-type assumption. However, as long as single crossing property holds, the same basic analysis using consistency constraints would apply with more preference types, and similar results are likely to be obtained.

Appendix A

Proofs of Lemmas

Lemma 6. *Suppose that Assumptions 1 and 2 hold. Then in any feasible allocation,* $q_1^L > q_1^H$ *and* $q_2^L < q_1^H$*.*

Proof. Suppose, by way of contradiction, that there exists a feasible allocation such that $q_{1L} \leq q_{1H}$ and $q_{2L} \geq q_{2H}$ where L is above and to the left of H. Then it must either bind at CC_H, or CC_L or not bind at all. To arrive at contradiction, we proceed by ruling out each case. Before moving on, note that the following inequality holds at an allocation such that $q_{1L} \leq q_{1H}$ and $q_{2L} \geq q_{2H}$.

$$\frac{u_L'(b_{2L}q_{2L})}{u_L'(b_{1L}q_{1L})} \leq \frac{u_H'(b_{2H}q_{2L})}{u_H'(b_{1H}q_{1L})} \leq \frac{u_H'(b_{2H}q_{2H})}{u_H'(b_{1H}q_{1H})} \tag{A1}$$

where the first inequality follows from the fact that the utility curve of L type is steeper than the curve for H type, and the second inequality follows from the fact that $\frac{q_{2L}}{q_{1L}} \geq \frac{q_{2H}}{q_{1H}}$ from the assumption that L is above and to the left of H. This inequality will come in handy when we derive contradiction for each case discussed below. Suppose first that CC_H binds. Once we rule out this case, the case where CC_L binds gets ruled out by symmetry. With CC_H binding, we have

$$u_H'(b_{1H}q_{1H})(b_1^H q_{1L} - b_1^H q_{1H}) + u_H'(b_{2H}q_{2H})(b_2^H q_{2L} - b_2^H q_{2H}) = 0$$

or

$$\frac{b_1^H(q_{1H}-q_{1L})}{b_2^H(q_{2H}-q_{2L})} = -\frac{u_H'(b_{2H}q_{2H})}{u_H'(b_{1H}q_{1H})}$$

Plugging it into CC_L, we have

$$\begin{aligned}
u_L'(b_{1L}q_{1L})(b_1^L q_{1H} - b_1^L q_{1L}) \leq -u_L'(b_{2L}q_{2L})(b_2^L q_{2H} - b_2^L q_{2L}) \\
u_L'(b_{1L}q_{1L})\frac{b_1^L}{b_2^L}\left[-\frac{b_2^H}{b_1^H}\frac{u_H'(b_{2H}q_{2H})}{u_H'(b_{1H}q_{1H})}\right] \geq -u_L'(b_{2L}q_{2L}) \\
\frac{b_2^H}{b_1^H}\frac{u_H'(b_{2H}q_{2H})}{u_H'(b_{1H}q_{1H})} \leq \frac{b_2^L}{b_1^L}\frac{u_L'(b_{2L}q_{2L})}{u_L'(b_{1L}q_{1L})} \qquad \text{(A2)}
\end{aligned}$$

Combining (A1) with $\frac{b_2^H}{b_1^H} > \frac{b_2^L}{b_1^L}$, we have

$$\frac{b_2^H}{b_1^H}\frac{u_H'(b_{2H}q_{2H})}{u_H'(b_{1H}q_{1H})} > \frac{b_2^L}{b_1^L}\frac{u_L'(b_{2L}q_{2L})}{u_L'(b_{1L}q_{1L})} \qquad \text{(A3)}$$

, which contradicts (A2). We checked that if L is above and to the left of H, then CC_H cannot bind. By a symmetric argument, CC_L cannot bind at such allocation.

So the only remaining possibility is the case where neither CC_H nor CC_L binds. Then

$$u_H'(b_{1H}q_{1H})(b_1^H q_{1L} - b_1^H q_{1H}) + u_H'(b_{2H}q_{2H})(b_2^H q_{2L} - b_2^H q_{2H}) < 0 \qquad \text{(A4)}$$

$$u_L'(b_{1L}q_{1L})(b_1^L q_{1H} - b_1^L q_{1L}) + u_L'(b_{2L}q_{2L})(b_2^L q_{2H} - b_2^L q_{2L}) < 0 \qquad \text{(A5)}$$

We have from (A4), $q_{1L} \leq q_{1H}$ and $q_{2L} \geq q_{2H}$,

$$\frac{q_1^H - q_1^L}{q_2^H - q_2^L} < -\frac{b_2^H}{b_1^H}\frac{u_H'(b_2^H q_2^H)}{u_H'(b_1^H q_1^H)}$$

and from (A5), $q_{1L} \leq q_{1H}$ and $q_{2L} \geq q_{2H}$,

$$\frac{q_1^H - q_1^L}{q_2^H - q_2^L} > -\frac{b_2^L}{b_1^L}\frac{u_L'(b_2^L q_2^L)}{u_L'(b_1^L q_1^L)}$$

which together gives

$$\frac{b_2^H}{b_1^H}\frac{u_H'(b_2^H q_2^H)}{u_H'(b_1^H q_1^H)} < \frac{q_1^H - q_1^L}{q_2^H - q_2^L} < \frac{b_2^L}{b_1^L}\frac{u_L'(b_2^L q_2^L)}{u_L'(b_1^L q_1^L)}$$

This also contradicts (A3). Therefore, all allocations such that $q_{1L} \leq q_{1H}$ and $q_{2L} \geq q_{2H}$ are not feasible, which means leads to the conclusion that any feasible allocation has $q_1^L > q_1^H$ and $q_2^L < q_1^H$. □

Now establish step 2 by ruling out H such that $\frac{q_{1H}}{q_{2H}} \geq r_H$. Let the Lagrangian for the problem be

$$\mathcal{L} = \lambda\Pi_H(Q_H) + (1-\lambda)\Pi_L(Q_L) + \mu\Lambda(Q_H, Q_L) + \zeta\Gamma(Q_L, Q_H) + \phi(U_H - \bar{U}_H) + \tau(U_L - \bar{U}_L)$$

where

$$\Lambda(Q_H, Q_L) = u_H'(b_{1H}q_{1H})(b_1^H q_1^H - b_1^H q_1^L) + u_H'(b_{2H}q_{2H})(b_2^H q_2^H - b_2^H q_2^L)$$

$$\Gamma(Q_L, Q_H) = u_L'(b_{1L}q_{1L})(b_1^L q_1^L - b_1^L q_1^H) + u_L'(b_{2L}q_{2L})(b_2^L q_2^L - b_2^L q_2^H)$$

λ is the fraction of H type in the population, and $\mu \geq 0, \zeta \geq 0, \phi \geq 0, \tau \geq 0$ are Lagrange multipliers associated with the profit function, CC_H, CC_L, MU_H, and MU_L respectively.

Define the following differential operators:

$$(D1)\nabla_{ICH} = \frac{\partial}{\partial q_1^H} - \frac{b_1^H}{b_2^H}\frac{u'(b_1^H q_1^H)}{u'(b_2^H q_2^H)}\frac{\partial}{\partial q_2^H}$$
$$(D2)\nabla_{ICL} = -\frac{\partial}{\partial q_1^L} + \frac{b_1^L}{b_2^L}\frac{u'(b_1^L q_1^L)}{u'(b_2^L q_2^L)}\frac{\partial}{\partial q_2^L}$$

Lemma 7. *In any optimal allocation in which CC_H binds, $\frac{q_2^H}{q_1^H} < r_H$. That is, H type's allocation is below and to the right of the undistorted ray.*

Proof. By applying (D1) to the Lagrange equation, we get

$$\nabla_{ICH}\mathcal{L} = \lambda\nabla_{ICH}\Pi_H + \mu\nabla_{ICH}\Lambda(Q_H, Q_L) + \zeta\nabla_{ICH}\Gamma(Q_L, Q_H) \tag{A6}$$

We can look at two possible cases, where CC_L does not bind, and where CC_L binds. In both cases, the proofs are similar and yield the same result that $\frac{q_2^H}{q_1^H} \geq r_H$ is not an optimum. We proceed by looking at the terms on RHS individually. First, $\lambda\nabla_{ICH}\Pi_H$ is weakly positive, by Lemma 5 and the assumption that $\frac{q_2^H}{q_1^H} \geq r_H$. Next we look at $\mu\nabla_{ICH}\Lambda(Q_H, Q_L)$. It can be proven that this term is always greater than zero.

$$\begin{aligned}
\nabla_{ICH}\Lambda(Q_H, Q_L) &= u_H''(b_1^H q_1^H)b_1^H(b_1^H q_1^H - b_1^H q_1^L) - u_H''(b_2^H q_2^H)b_2^H(b_2^H q_2^H - b_2^H q_2^L)\frac{b_1^H u_H'(b_1^H q_1^H)}{b_2^H u_H'(b_2^H q_2^H)} \\
&\quad + b_1^H u_H'(b_1^H q_1^H) - b_2^H u_H'(b_2^H q_2^H)\frac{b_1^H u_H'(b_1^H q_1^H)}{b_2^H u_H'(b_2^H q_2^H)} \\
&= b_1^H u_H''(b_1^H q_1^H)(b_1^H q_1^H - b_1^H q_1^L) - b_2^H u_H''(b_2^H q_2^H)(b_2^H q_2^H - b_2^H q_2^L)\frac{b_1^H u_H'(b_1^H q_1^H)}{b_2^H u_H'(b_2^H q_2^H)} \\
&= b_1^H u_H'(b_1^H q_1^H)\left[\frac{u_H''(b_1^H q_1^H)}{u_H'(b_1^H q_1^H)}(b_1^H q_1^H - b_1^H q_1^L) - \frac{u_H''(b_2^H q_2^H)}{u_H'(b_2^H q_2^H)}(b_2^H q_2^H - b_2^H q_2^L)\right]
\end{aligned}$$

Since u is strictly concave in q, $u' > 0$ and $u'' < 0$. Thus we have

$$\frac{u_H''(b_1^H q_1^H)}{u_H'(b_1^H q_1^H)} < 0 \quad \text{and} \quad \frac{u_H''(b_2^H q_2^H)}{u_H'(b_2^H q_2^H)} < 0 \tag{A7}$$

By Lemma 6, $q_1^H - q_1^L < 0$ and $q_2^H - q_2^L > 0$, which gives $\nabla_{ICH}\Lambda(Q_H, Q_L) > 0$.

Next, we look at the third term $\zeta\nabla_{ICH}\Gamma(Q_L, Q_H)$. If CC_L does not bind, $\zeta = 0$, and it immediately follows that $\nabla_{ICH}\mathcal{L} > 0$, proving that allocation for H type such that $\frac{q_2^H}{q_1^H} \geq r_H$ is not an optimum. Consider instead the case where CC_L binds. Then

$\zeta \neq 0$ and we need to look at $\nabla_{ICH}\Gamma(Q_L, Q_H)$.

$$\nabla_{ICH}\Gamma(Q_L, Q_H) = -b_1^L u_L'(b_1^L q_1^L) + b_2^L u_L'(b_2^L q_2^L) \frac{b_1^H u_H'(b_1^H q_1^H)}{b_2^H u_H'(b_2^H q_2^H)}$$
$$= b_2^L u_L'(b_2^L q_2^L) \left[-\frac{b_1^L u_L'(b_1^L q_1^L)}{b_2^L u_L'(b_2^L q_2^L)} + \frac{b_1^H u_H'(b_1^H q_1^H)}{b_2^H u_H'(b_2^H q_2^H)} \right] = 0$$

The last equality follows from both CC_H and CC_L binding, so that $\frac{b_1^L u_L'(b_1^L q_1^L)}{b_2^L u_L'(b_2^L q_2^L)} = \frac{b_1^H u_H'(b_1^H q_1^H)}{b_2^H u_H'(b_2^H q_2^H)}$. Thus $\nabla_{ICH}\Gamma(Q_L, Q_H) = 0$. This establishes that allocations where $\frac{q_2^H}{q_1^H} \geq r_H$ is not an optimum in general. □

By Lemma 7, we have ruled out this candidate for H type allocation where $\frac{q_2^H}{q_1^H} \geq r_H$. Next, consider cases where $\frac{q_2^H}{q_1^H} < r_H$. We proceed in two steps. First, we establish that CC_L cannot bind when $\frac{q_2^L}{q_1^L} \geq r_L$(Step 3). And then we rule out the case where CC_L binds at $\frac{q_2^L}{q_1^L} < r_L$ (Step 4). We first establish the first half of Step 3:

Lemma 8. *If CC_H binds at an optimal allocation and $\frac{q_2^L}{q_1^L} \geq r_L$, then CC_L cannot bind.*

Proof. Let us suppose, by way of contradiction, that both CC_H and CC_L bind, such that

$$\frac{b_1^L u_L'(b_1^L q_1^L)}{b_2^L u_L'(b_2^L q_2^L)} = \frac{b_1^H u_H'(b_1^H q_1^H)}{b_2^H u_H'(b_2^H q_2^H)}.$$

Let $r_L = \frac{q_2^{L*}}{q_1^{L*}}$, and $r_H = \frac{q_2^{H*}}{q_1^{H*}}$. Then $\frac{q_2^L}{q_1^L} \geq \frac{q_2^{L*}}{q_1^{L*}}$, and since u_L is concave, $u'(x)$ is decreasing in x, and we have

$$\frac{b_1^L u_L'(b_1^L q_1^L)}{b_2^L u_L'(b_2^L q_2^L)} \geq \frac{b_1^L u_L'(b_1^L q_1^{L*})}{b_2^L u_L'(b_2^L q_2^{L*})}.$$

Similarly, $\frac{q_2^H}{q_1^H} < \frac{q_2^{H*}}{q_1^{H*}}$, and

$$\frac{b_1^H u_H'(b_1^H q_1^H)}{b_2^H u_H'(b_2^H q_2^H)} < \frac{b_1^H u_H'(b_1^H q_1^{H*})}{b_2^H u_H'(b_2^H q_2^{H*})}.$$

Therefore,

$$\frac{b_1^L u_L'(b_1^L q_1^{L*})}{b_2^L u_L'(b_2^L q_2^{L*})} < \frac{b_1^H u_H'(b_1^H q_1^{H*})}{b_2^H u_H'(b_2^H q_2^{H*})},$$

which is a contradiction, since by assumption on preferences, $\frac{b_1^H}{b_2^H} < \frac{b_1^L}{b_2^L}$, and $\frac{q_1^{H*}}{q_2^{H*}} > \frac{q_1^{L*}}{q_2^{L*}}$. □

Corollary 1. *In any optimal allocation in which CC_H binds, $\frac{q_2^L}{q_1^L} < r_L$. That is, L's allocation is below and to the right of undistorted ray.*

Proof. By Lemma 8, we established that if CC_H binds at an optimal allocation and $\frac{q_2^L}{q_1^L} \geq r_L$, then CC_L cannot bind. We can further rule out the case in which CC_L does not bind to establish the corollary. First order condition is given by

$$-\nabla_{ICL}\mathcal{L} = -\left[(1-\lambda)\nabla_{ICL}\Pi_L + \zeta\nabla_{ICL}\Gamma(Q_L, Q_H) + \mu\nabla_{ICL}\Lambda(Q_H, Q_L)\right]$$

$-(1-\lambda)\nabla_{ICL}\Pi_L$ is weakly positive by Lemma 5, and the assumption that $\frac{q_2^L}{q_1^L} \geq r_L$. $\zeta = 0$ since CC_L is slack. CC_H binding indicates, $\mu \neq 0$, while the movement makes $\Lambda(Q_H, Q_L) > 0$. Thus, the first order condition does not hold, so such allocation cannot be an optimum. □

Lemma 9. *If CC_H binds and $\frac{q_2^H}{q_1^H} < r_H$, then CC_L cannot bind in allocations where $\frac{q_2^L}{q_1^L} < r_L$.*

Proof. Suppose, by way of contradiction, that CC_H binds at $\frac{q_2^H}{q_1^H} < r_H$ and CC_L also binds at $\frac{q_2^L}{q_1^L} < r_L$. Then we can suspect that a movement of L's allocation along its indifference curve towards the first best allocation would improve profits. The first order condition that reflects this idea is

$$\nabla_{ICL}\mathcal{L} = (1-\lambda)\nabla_{ICL}\Pi_L + \zeta\nabla_{ICL}\Gamma(Q_L, Q_H) + \mu\nabla_{ICL}\Lambda(Q_H, Q_L).$$

The first term on the right hand side is positive by Lemma 5 and the assumption that $\frac{q_2^L}{q_1^L} < r_L$, and the third term is zero in the first order. It is then sufficient to show that the second term is positive (non-negative) to establish that such allocation cannot be an optimum.

$$\nabla_{ICL}\Gamma(Q_L,Q_H) = \left(-\frac{\partial}{\partial q_1^L} + \frac{b_1^L u_L'(b_1^L q_1^L)}{b_2^L u_L'(b_2^L q_2^L)}\frac{\partial}{\partial q_2^L}\right)\left(u_L'(b_1^L q_1^L)(b_1^L q_1^L - b_1^L q_1^H) + u_L'(b_2^L q_2^L)(b_2^L q_2^L - b_2^L q_2^H)\right))$$

$$= -b_1^L u''(b_1^L q_1^L)(b_1^L q_1^L - b_1^L q_1^H) + b_2^L u''(b_2^L q_2^L)(b_2^L q_2^L - b_2^L q_2^H)\frac{b_1^L u'(b_1^L q_1^L)}{b_2^L u'(b_2^L q_2^L)}$$

$$-b_1^L u'(b_1^L q_1^L) + b_2^L u'(b_2^L q_2^L)\frac{b_1^L u'(b_1^L q_1^L)}{b_2^L u'(b_2^L q_2^L)}$$

$$= b_1^L u'(b_1^L q_1^L)\left[-\frac{u''(b_1^L q_1^L)}{u'(b_1^L q_1^L)}(b_1^L q_1^L - b_1^L q_1^H) + \frac{u''(b_2^L q_2^L)}{u'(b_2^L q_2^L)}(b_2^L q_2^L - b_2^L q_2^H)\right]$$

From (A7) and Lemma 6, we have $\nabla_{ICL}\Gamma(Q_L,Q_H) > 0$. Therefore CC_L cannot bind at allocations where $\frac{q_2^L}{q_1^L} < r_L$. □

Lemma 10. *With linear pricing, utility maximization problem can be reformulated with indirect utilities of iso-n's if preferences are homothetic.*

Proof. Suppose u is homothetic. Then there exists a function h such that

$$u(nq_1, nq_2)) = h(F(nq_1, nq_2)) \tag{11}$$

where $F(nq_1, nq_2)$ is a homogeneous of degree 1 function. Then

$$\frac{\partial u}{\partial n} = h'(nF(q_1,q_2))F(q_1,q_2) = 1$$

$$h'(h^{-1}(V^*))(h^{-1}(V^*)) = n$$

Where $V^*(q_1,q_2) \equiv u(n^*(q_1,q_2)q_1, n^*(q_1,q_2)q_2)$. Then it follows that points on the same indifference curve in (q_1,q_2) space are also points of equal n^*. □

Bibliography

[1] Armstrong, M., and Rochet, J. (1999): "Multi-dimensional Screening: A User's Guide," *European Economic Review*, 43, 959-979.

[2] Mussa M., and Rosen S. (1978): "Monopoly and Product Quality," *Journal of Economic Theory*, 18, 301-317.

[3] Myerson, R. (1979): "Incentive Compatibility and the Bargaining Problem," *Econometrica*, 47, 61-74.

[4] Rothschild, C. (2013): "Non-Exclusivity, Linear Pricing, and Annuity Market Screening," *Journal of Risk and Insurance*, Forthcoming.

[5] Stiglitz, J. (1977): "Monopoly, Non-Linear Pricing and Imperfect Information: The Insurance Market," *The Review of Economic Studies*, 44, 407-430.