2014

Khovanov Homology and Calculus of Functors

Farrah Yhee
fyhee@wellesley.edu

Khovanov Homology and Calculus of Functors

Author: **Farrah Yhee**
Advisor: Ismar Volić

Submitted in Partial Fulfillment of the
Prerequisites for Honors
in the Department of Mathematics

May 15, 2014
Wellesley College

Contents

Abstract

A central question in knot theory is the classification of knots. Given two knots, how can we determine if they are different or the same? To answer this question, we develop and study knot invariants which are properties of knots that remain unchanged under isotopy. Khovanov homology is a powerful knot invariant that is able to distinguish many knots. However, because it is constructed in a combinatorial and algebraic manner, Khovanov homology lacks any geometric or topological motivation. Since Khovanov homology encodes information about topological objects, it would be ideal if it could be interpreted from a topological perspective. One way to approach this is through the lens of manifold calculus of functors and more specifically, the Taylor tower for spaces of long knots. Recent developments have shown that the Jones polynomial, another knot invariant, is encoded in the Taylor tower for knots. Since the Jones polynomial can be extracted from Khovanov homology, it is natural to ask if the Taylor tower can provide a space level realization of Khovanov homology. This paper provides an introduction to Khovanov homology and calculus of functors and offers conjectures that relate the two notions.

Acknowledgements

I would like to thank the following individuals for their support during the thesis process:

To Professor Ismar Volić: thank you for your willingness to take me on as a thesis student despite being on sabbatical, for your abundant patience, enthusiasm and belief in me, and for all your support, encouragement, and advice about mathematics these past two years.

To Young-ah Lucy Lee, my fellow thesis buddy: thank you for suffering with me, for the late night conversations, for your encouragement and forward-looking practicality when I lacked the confidence to pursue mathematics.

To Maria Jun and Anastasia Hou: thank you both for your continuous stream of late night snacks and your willingness to listen to my rather incoherent conversations about circles and arrows and knots. Anastasia, thank you for all the careful editing and for all your love and care. Maria, thank you for all your prayers.

To Anubhav Sinha: thank you for your all your help with LaTeX, your careful edits, the long philosophical conversations about mathematics, science, life, etc. and for your firm belief in my abilities.

Finally, I would like to thank my family: Mom, Dad, Jiwon, and Sang, thank you for all your love and support these past 22 years. Despite your unfamiliarity with mathematics, you have always had the utmost confidence and belief in me. Thank you for all your encouragement and prayers.

Introduction and Motivation

Intuitively, a mathematical knot results from taking a string, tangling it and then gluing the ends together. Two knots are equivalent if one knot can be deformed into the other without breaking the initial knot open. More formally, a knot is an embedding of S^1 into $\mathbb{R}^3$. Two knots are equivalent if there is a deformation, more precisely isotopy, from one to the other.

A central question in knot theory is the classification of knots. That is, given two knots, how can we determine if they are distinct or the same? Knot invariants are tools that allow us to address this question. A knot invariant is a function from the collection of knots to a collection of mathematical objects such as integers or polynomials that assigns to each knot an object in a manner that is invariant under isotopy. This means that a knot invariant will assign the same object to isotopic knots. Hence, if an invariant assigns different objects to two knots, then the knots are different. It is important to note, however, that the converse is not true. An invariant may assign to two different knots the same object. We say that an invariant is stronger than another invariant if it can distinguish more distinct knots.

Two powerful knot invariants are the Jones polynomial and Khovanov homology. The Jones polynomial associates to each knot a polynomial constructed from certain geometric resolutions of the knot while the Khovanov homology associates a homology constructed from the same resolutions to each knot. They provide novel ways of analyzing knots and are very successful in disguishing knots. For example, both invariants can distinguish between the right-hand and left-hand trefoils - knots that many other invariants cannot distinguish. This paper will explore Khovanov homology, which is the more powerful of the two.

Khovanov homology is considered the "categorification of the Jones polynomial" and encodes all the data that the Jones polynomial captures and more. It is strictly stronger than the Jones polynomial in the sense that Khovanov homology distinguishes all the knots that Jones polynomial can distinguish and other knots that the Jones polynomial cannot. While the Jones polynomial cannot distinguish the knots 5_1 and 10_{132} or the knot 9_{42} and its mirror, Khovanov homology can distinguish them. Furthermore, it was recently proven that Khovanov homology is an unknot detector whereas it is unknown whether or not the Jones polynomial can detect the unknot.

Despite its usefulness as an invariant, the current understanding of Khovanov homology is mostly combinatorial and algebraic. Khovanov homology lacks a satisfying geometric or topological motivation. In particular, what geometric and topological properties of a knot does Khovanov homology capture that gives it its invariance and its ability to distinguish knots? Because knots are topological objects, interpreting Khovanov homology as a topological construction would be ideal. One way to approach this problem is to study Khovanov homology through the lens of manifold calculus of functors. To do this, we will need the language of category theory.

Category theory captures universal mathematical properties that are found in diverse fields of mathematics. Categories and functors generalize the notions of sets and functions. A category is a collection consisting of mathematical objects and sets of maps between these objects. Examples of categories include the category of abelian groups and the category of topological spaces. A functor is a map relating different categories. Manifold calculus of functors studies certain functors between a poset of open subsets of a smooth closed manifold M to the category of topological spaces. It is mostly used in homotopy theory but has been shown to have applications in knot theory. For more explicit examples, see [Vol]. In this context, we can build what is called the Taylor tower for spaces of long knots.

The idea of a Taylor tower is analogous to that of a Taylor series for a smooth function f about a point x. Just as the behavior of f about x can be approximated by a sequence of simpler polynomial functions, the behavior of a functor $F : \mathcal{C} \longrightarrow \mathcal{C}'$ can be approximated by a sequence of simpler functors $T_k F : \mathcal{C} \longrightarrow \mathcal{C}'$. We call $T_k F$ the k-th stage of the Taylor tower. The Taylor tower is a "categorification of the Taylor series." In the case of knots, we can construct a Taylor tower for the space of knots $\mathcal{K}^3$ which captures the isotopy classes of knots and knot invariants. Understanding how Khovanov homology is encoded in this construction would yield a space level realization of the knot invariant.

This paper provides an introduction to Khovanov homology and manifold calculus of functors in the case of knots. In **Section 1**, we review some basic definitions from category theory as well as important topological objects that will be used throughout the paper. **Section 2** introduces and builds the theory of cubical diagrams and homotopy limits of punctured cubes of spaces. **Section 3** reviews the construction of Khovanov homology and provides a brief sketch of Everitt and Turner's homotopy theoretic interpretation of Khovanov homology from [EvT]. Everitt and Turner's interpretation yields a space level construction of Khovanov homology and shares similarities to the construction of the Taylor tower for spaces of long knots. We elucidate this relationship, by proving that their interpretation is equivalent to the total fiber of a cubical diagram of spaces, which is itself a homotopy limit. In **Section 4**, we construct the Taylor tower of long knots where each stage of the tower is a homotopy limit of a punctured cubical diagram of spaces and provide equivalent constructions that are more often used in practice. Finally, in **Section 5**, we provide conjectures connecting Khovanov homology with the Taylor tower for spaces of long knots.

1 Preliminaries

Definition 1.1. A **category** $\mathcal{C}$ consists of:

- a collection of **objects** $\mathrm{Ob}(\mathcal{C})$
- a set of **morphisms** $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ for each $X, Y \in \mathcal{C}$
- a **law of composition** $\circ : \mathrm{Hom}_{\mathcal{C}}(X, Y) \times \mathrm{Hom}_{\mathcal{C}}(Y, Z) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(X, Z)$ that is associative, i.e. $\alpha \circ (\beta \circ \gamma) = (\alpha \circ \beta) \circ \gamma$

- for each $X \in \mathcal{C}$, an **identity morphism** $\mathbf{1}_X \in \mathrm{Hom}_{\mathcal{C}}(X,X)$ such that $f \circ 1_X = 1_X \circ f = f$ for any $f \in \mathrm{Hom}_{\mathcal{C}}(X,X)$.

Example 1.2. The category **Set** has sets for objects and functions for morphisms.

Example 1.3. The category of topological spaces, denoted **Top**, has topological spaces for objects and continuous maps for morphisms. Similarly, the category of based topological spaces, denoted $\mathbf{Top}_*$ has based topological spaces for objects and based continuous maps for morphisms. Note that a based continuous map $f : X \longrightarrow Y$ sends the basepoint of X to the basepoint of Y.

Definition 1.4. A **small category** $\mathcal{C}$ is a category such that $\mathrm{Ob}(\mathcal{C})$ is a set.

Example 1.5. Any group G is a small category if we consider the group acting on itself by left multiplication. The object is the group itself and morphisms elements of G with compositions compatible with the group operation.

Definition 1.6. Given two categories $\mathcal{C}$ and $\mathcal{C}'$, a **(covariant) functor** $F : \mathcal{C} \longrightarrow \mathcal{C}'$ is a map that associates to each object $X \in \mathcal{C}$ an object $F(X) \in \mathcal{C}'$ and to each morphism $\alpha : X \longrightarrow Y$ in $\mathcal{C}$ a morphism $F(\alpha) : F(X) \longrightarrow F(Y)$ in $\mathcal{C}'$ such that

- $F(\mathbf{1}_X) = \mathbf{1}_{F(X)}$ for all $X \in \mathcal{C}$.
- $F(\alpha \circ \beta) = F(\alpha) \circ F(\beta)$ for $\alpha \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$ and $\beta \in \mathrm{Hom}_{\mathcal{C}}(Y,Z)$

Definition 1.7. Let $\mathcal{C}$ be a category. Then the **opposite category** of $\mathcal{C}$, denoted $\mathcal{C}^{\mathbf{op}}$, is the category with objects the same as $\mathcal{C}$ and morphisms defined by

$$\mathrm{Hom}_{\mathcal{C}^{op}}(X,Y) = \mathrm{Hom}_{\mathcal{C}}(Y,X).$$

Definition 1.8. Let $\mathcal{C}$ and $\mathcal{C}'$ be categories. We say that F is a contravariant functor if it is a functor from $\mathcal{C}^{op}$ to $\mathcal{C}'$. Thus F assigns a morphism $F(Y) \longrightarrow F(X)$ to a morphism $X \longrightarrow Y$.

Definition 1.9. Let $F, G : \mathcal{C} \longrightarrow \mathcal{D}$ be functors. A **natural transformation** η associates to each $X \in \mathrm{Ob}(\mathcal{C})$ a morphism $\eta_X : F(X) \longrightarrow G(X)$ such that for every morphism $f : X \longrightarrow Y$, we have $\eta_Y \circ F(f) = G(f) \circ \eta_X$, i.e. the following diagram commutes:

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\eta_X"'] & F(Y) \arrow[d, "\eta_Y"] \\ G(X) \arrow[r, "G(f)"'] & G(Y) \end{tikzcd}$$

Definition 1.10. Let X and Y be topological spaces and let $\mathrm{Map}(X,Y)$ be the space of continuous functions between X and Y. Then $\mathrm{Map}(X,Y)$ is topologized as follows: Given a compact subset K of X and an open subset U of Y, let $U^K = \{f \in \mathrm{Map}(X,Y) : f(K) \subset U\}$. Then the sets U^K form a subbase of $\mathrm{Map}(X,Y)$. This is called the **compact-open** topology.

Definition 1.11. If $A \subset X$ and $B \subset Y$, then $\mathrm{Map}((X,A),(Y,B))$ is the space of continuous maps of pairs whose elements are continuous maps f such that $f(A) \subset B$. It is topologized as a subspace of $\mathrm{Map}(X,Y)$.

If X and Y are based spaces with basepoints A and B respectively, we write $\mathrm{Map}_*(X,Y)$.

Example 1.12. An important example that will be revisited throughout this paper is the *loop space* of a based topological space Y with basepoint $y \in Y$. It is defined as $\mathrm{Map}((I,\partial I),(Y,y))$ and has a basepoint, the constant map c_y which sends I to y. This space is denoted ΩY. Since $I/\partial I \cong S^1$, it follows that $\Omega Y \cong \mathrm{Map}_*(S^1,Y)$.

The *n-fold loop space* is defined inductively as $\Omega^n Y = \Omega\Omega^{n-1}Y$. We will use the fact that $\Omega^n Y \cong \mathrm{Map}_*(S^n,Y)$ in later examples.

2 Cubical Diagrams and Homotopy Limits

2.1 Cubical Diagrams

Definition 2.1. A **diagram** in category $\mathcal{C}$ is a functor F from a small category $\mathcal{I}$, called the indexing category, to $\mathcal{C}$.

The small category $\mathcal{I}$ serves to index the image of F in a manner that captures the shape of image of F. The diagrams we are interested in are n-cubical diagrams and in particular, punctured n-cubical diagrams. The indexing category that we will use exclusively are posets of subsets of $\{1,2,\ldots,n\}$ ordered by inclusion.

Definition 2.2. Let $\underline{n} = \{1,2,\ldots,n\}$. Then the **cubical indexing category** $\mathcal{P}(\underline{n})$ is the category with the subsets of $\underline{n}$ for objects and inclusions for morphisms.

Example 2.3. Pictorially $\mathcal{P}(\underline{1})$ is

$$\emptyset \longrightarrow \{1\}$$

and $\mathcal{P}(\underline{2})$ is

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] & \{1\} \arrow[d] \\
\{2\} \arrow[r] & \{1,2\}
\end{tikzcd}$$

and $\mathcal{P}(\underline{3})$ is

$$\begin{tikzcd}
\emptyset \arrow[rr] \arrow[dr] \arrow[dd] & & \{1\} \arrow[dr] \arrow[dd] & \\
& \{2\} \arrow[rr] \arrow[dd] & & \{1,2\} \arrow[dd] \\
\{3\} \arrow[rr] \arrow[dr] & & \{1,3\} \arrow[dr] & \\
& \{2,3\} \arrow[rr] & & \{1,2,3\}.
\end{tikzcd}$$

Definition 2.4. Let $\mathcal{P}_0(\underline{n})$ be the poset of non-empty subsets of $\underline{n}$. Then $\mathcal{P}_0(\underline{n})$ is the n-cube $\mathcal{P}(\underline{n})$ with the empty set removed and is called a **punctured indexing category.**

Example 2.5. We can view $\mathcal{P}_0(\underline{3})$ as

$$\begin{tikzcd}
& & \{1\} \arrow[dr] \arrow[dd] & \\
& \{2\} \arrow[rr] \arrow[dd] & & \{1,2\} \arrow[dd] \\
\{3\} \arrow[rr] \arrow[dr] & & \{1,3\} \arrow[dr] & \\
& \{2,3\} \arrow[rr] & & \{1,2,3\}.
\end{tikzcd}$$

Furthermore, any punctured n-cube can be redrawn as a barymetrically subdivided $(n-1)$ simplex. For example $\mathcal{P}_0(\underline{2})$ is

$$\{1\} \longrightarrow \{1,2\} \longleftarrow \{2\}$$

and $\mathcal{P}_0(\underline{3})$ is

$$\begin{tikzcd}
& & \{1\} \arrow[dl] \arrow[dr] & & \\
& \{1,3\} \arrow[r] & \{1,2,3\} & \{1,2\} \arrow[l] & \\
\{3\} \arrow[ur] \arrow[rr] & & \{2,3\} \arrow[u] & & \{2\} \arrow[ll] \arrow[ul]
\end{tikzcd}$$

Definition 2.6.

- An **n-cubical diagram** or an **n-cube** in $\mathcal{C}$ is a functor $\mathcal{X} : \mathcal{P}(\underline{n}) \longrightarrow \mathcal{C}$.
- A **punctured n-cubical diagram** or a **punctured n-cube** is a functor $\mathcal{X} : \mathcal{P}_0(\underline{n}) \longrightarrow \mathcal{C}$.

Notationally, for every $S \in \mathcal{I}$, we let $\mathcal{X}(S) = X_S$. For example $\mathcal{X}(\emptyset) = X_\emptyset$ and $\mathcal{X}(\{1,2,3\}) = X_{\{1,2,3\}}$. We further suppress the notation by dropping the set notation so that $X_{\{1,2,3\}} =$

X_{123}. Similarly, each morphism $\mathcal{X}_S \longrightarrow \mathcal{X}_T$ is denoted $X(S \longrightarrow T)$. Then a 3-cubical diagram can be represented as a commutative cube $\mathcal{X}(\mathcal{P}(\underline{3}))$ which is illustrated as

$$\begin{tikzcd}
X_\emptyset \arrow[rr] \arrow[dr] \arrow[dd] & & X_1 \arrow[dr] \arrow[dd] & \\
& X_2 \arrow[rr, crossing over] & & X_{12} \arrow[dd] \\
X_3 \arrow[rr] \arrow[dr] & & X_{13} \arrow[dr] & \\
& X_{23} \arrow[rr] \arrow[from=uu, crossing over] & & X_{123}
\end{tikzcd}$$

and a punctured 3-cubical diagram, represented by $\mathcal{X}(\mathcal{P}_0(\underline{3}))$, is illustrated as

$$\begin{tikzcd}
& & X_1 \arrow[dl] \arrow[dr] & & \\
& X_{13} \arrow[r] & X_{123} & X_{12} \arrow[l] & \\
X_3 \arrow[ur] \arrow[rr] & & X_{23} \arrow[u] & & X_2 \arrow[ul] \arrow[ll]
\end{tikzcd}$$

.

Remark 2.7. For every $n > 1$, an n-cube can be viewed as a natural transformation between $(n-1)$-cubes in n distinct ways.

2.2 Homotopy Limits of Punctured Cubes of Spaces

Limits of diagrams in the category of spaces satisfy a universal property and capture standard constructions such as products, inverse limits and gcd. However, they are not well behaved under homotopy. For example, given two cubical diagrams $\mathbf{B}$ and $\mathbf{B}'$ such that each space X_S in $\mathbf{B}$ is homotopy equivalent to the corresponding space X'_S in $\mathbf{B}'$, it is not necessarily true that the limit of $\mathbf{B}$ is homotopy equivalent to the limit of $\mathbf{B}'$. Consider the following definition and example:

Definition 2.8. The limit of a punctured 2-cube $X \xrightarrow{f} Z \xleftarrow{g} Y$ is the subspace of $X \times Y$ consisting of

$$\{(x, y) : f(x) = g(y)\}$$

.
This is also called the **pullback** of $X \xrightarrow{f} Z \xleftarrow{g} Y$.

Example 2.9. Now consider the punctured 2-cube $S_0 = \{0\} \longrightarrow [0,1] \longleftarrow \{1\}$ where the maps are inclusions of the points $\{0\}$ and $\{1\}$ to the endpoints of $[0,1]$. The pullback of S_0 is $\emptyset$. However, the pullback of $\{0\} \longrightarrow * \longleftarrow \{1\}$ is a single point even though $[0,1] \simeq *$.

The notion of **homotopy limits** addresses this issue. Homotopy limits are essentially formed by "fattening up" limits to form new objects that allow for homotopy invariance.

Although there is a general notion of a homotopy limit over categories, we will restrict our attention to cubical diagrams and the case where $\mathcal{C} = \textbf{Top}$. Furthermore, we will focus on homotopy limits of punctured cubes. This is because a non-punctured cube has an **initial object**. An initial object of a category $\mathcal{C}$ is an object X such that for every object $Y \in \mathcal{C}$, there is a unique morphism $X \longrightarrow Y$. The homotopy limit of an n-cube is homotopy equivalent to its initial space, which is $X_\emptyset$. The homotopy limits of punctured n-cubes turn out to be more interesting. We will, however, use the homotopy limit of a 1-cube $X \longrightarrow Y$ when defining the **total fiber** of an n-cube. To this end, the definition of the homotopy limit of a 1-cube is given below:

Definition 2.10. Let $X_\emptyset$ be topological space and X_1 be a based topological space with basepoint a. Let $f : X_\emptyset \longrightarrow X_1$ be a map. Then $\mathrm{holim}(X_\emptyset \xrightarrow{f} X_1)$ is the subspace

$$\{(x, \alpha) : x \in X_\emptyset, \alpha : I \longrightarrow X_1, \alpha(0) = f(x), \alpha(1) = a\}$$

of $X_\emptyset \times \mathrm{Map}(I, Y)$. This is called the **homotopy fiber** of f over $a \in X_1$ and it is denoted $\mathrm{hofiber}_a(f)$.

Example 2.11. Let $f : a \hookrightarrow Y$ be the inclusion of the basepoint a of Y into Y. Then

$$\mathrm{hofiber}_a(f) = \Omega Y,$$

the based loop space of Y.

We will now define homotopy limits for punctured n-cubes:

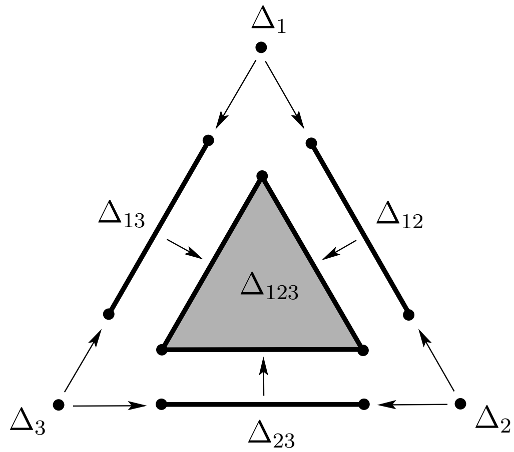
Definition 2.12. Define $\Delta(\bullet) : \mathcal{P}_0(\underline{n}) \longrightarrow \mathrm{Top}$ by

$$\Delta(S) = \left\{(t_1, ..., t_n) : 0 \leq t_i \leq 1, \sum t_i = 1, t_i = 0 \text{ for } i \notin S\right\}.$$

and each inclusion in $\mathcal{P}_0(\underline{n})$ is mapped to an inclusion in Top, i.e. the 0-simplices $\Delta(1)$ and $\Delta(2)$ map via inclusion to the endpoints of the 1-simplex $\Delta(12)$ and similarly for the rest of the diagram.

This is called the **punctured n-cube of simplices**. Note that $\Delta(S) \cong \Delta^{|S|-1}$, the simplex of dimension $|S| - 1$.

Example 2.13. $\Delta(\bullet) : \mathcal{P}_0(\underline{3}) \longrightarrow \mathrm{Top}$ is



Definition 2.14. Let $\mathcal{X}$ and $\mathcal{Y}$ be n-cubes. We can consider the **space of natural transformation** between $\mathcal{X}$ and $\mathcal{Y}$, denoted $\mathrm{Nat}(\mathcal{X}, \mathcal{Y})$. It consists of collections of maps $(f_S)_{S \in \mathcal{P}(\underline{n})}$ such that each collection determines a natural transformation between $\mathcal{X}$ and $\mathcal{Y}$ and is topologized as a subspace of the product space $\prod_{S \in \mathcal{P}(\underline{n})} \mathrm{Map}(X_S, Y_S)$ where each $\mathrm{Map}(X_S, Y_S)$ is topologized with the compact-open topology described in Definition 1.10.

Definition 2.15. Let $\mathcal{X} : \mathcal{P}_0(\underline{n}) \longrightarrow \mathrm{Top}$. The **homotopy limit of** $\mathcal{X}$ is defined as

$$\mathrm{holim}_{\mathcal{P}_0(\underline{n})} \mathcal{X} = \mathrm{Nat}_{\mathcal{P}_0(\underline{n})}(\Delta(\bullet), \mathcal{X}(\bullet)).$$

Remark 2.16. At times it will be notationally convenient to denote $\mathrm{holim}_{\mathcal{P}_0(\underline{n})} \mathcal{X}$ as $\mathrm{holim}_{S \in \mathcal{P}_0(\underline{n})} X_S$.

Example 2.17. Let X_1 be the punctured 1-cube. A point in $\mathrm{holim}(X_1)$ consists of a map from $*$ to a point $x \in X_1$, which can be considered as just the point $x \in X_1$. Then $\mathrm{holim}(X_1) = \mathrm{Nat}(*, X_1) \cong X_1$.

Example 2.18. Let $\mathcal{X} : \mathcal{P}_0(\underline{2}) \longrightarrow \mathbf{Top}$ be the punctured 2-cube $X_1 \xrightarrow{f} X_{12} \xleftarrow{g} X_2$. Then $\mathrm{holim}(X_1 \xrightarrow{f} X_{12} \xleftarrow{g} X_2) = \mathrm{Nat}(* \to I \leftarrow *, X_1 \xrightarrow{f} X_{12} \xleftarrow{g} X_2)$. This is the subspace

$$\{(x, \alpha, y) : x \in X_1, y \in X_2, \alpha \in \mathrm{Map}(I, X_{12}) \text{ s.t. } \alpha(0) = f(x), \alpha(1) = g(y)\}$$

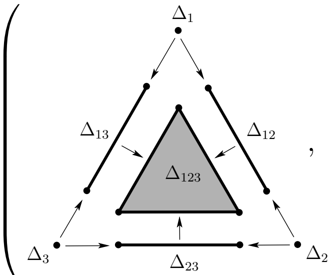
of $X_1 \times \mathrm{Map}(I, X_{12}) \times X_1$. This is also called the **homotopy pullback** of $X_1 \xrightarrow{f} X_{12} \xleftarrow{g} X_2$.

Example 2.19. Consider the punctured 2-cube $* \longrightarrow X \longleftarrow *$ where X is a based space with basepoint $*$ and the maps are inclusions of the basepoint. A point in the homotopy pullback is $(*, \alpha, *)$, where α is a path beginning and ending at $*$. Thus the homotopy pullback consists of loops with basepoint $*$, i.e. $\mathrm{holim}(* \longrightarrow X \longleftarrow *) \simeq \Omega X$, the loopspace of X.

Example 2.20. Consider the punctured 2-cube $\mathcal{X} = X \xrightarrow{f} Y \xleftarrow{g} *$. A point in $\mathrm{holim}(\mathcal{X})$ looks like $(x, \alpha, *)$ where $x \in X$ and α is a path between $f(x)$ and $g(*)$. If we choose $g(*)$ to be the basepoint of Y, then a point in $\mathrm{holim}(\mathcal{X})$ is precisely a point in $\mathrm{hofiber}_{g(*)}(X \longrightarrow Y)$, i.e. $\mathrm{holim}(\mathcal{X}) \cong \mathrm{hofiber}_{g(*)}(X \longrightarrow Y)$.

Example 2.21. Let $\mathcal{X} : \mathcal{P}_0(\underline{n}) \longrightarrow \text{Top}$ be a punctured 3-cube. Then $\operatorname{holim}_{P_0(\underline{3})}(\mathcal{X})$ is

Nat (



,

$$\begin{tikzcd} & X_1 \arrow[dl] \arrow[dr] & \\ X_{13} \arrow[r] & X_{123} & X_{12} \arrow[l] \\ X_3 \arrow[u] \arrow[r] & X_{23} \arrow[u] & X_2 \arrow[u] \arrow[l] \end{tikzcd}$$

).

A point in $\operatorname{holim}_{P_0(\underline{3})}(\mathcal{X})$ consists of:

- points in X_1, X_2, X_3
- paths $\alpha_{12}, \alpha_{13}, \alpha_{23}$ in X_{12}, X_{13}, X_{23} respectively
- a two parameter path α_{123} in X_{123}

such that they are all compatible with the maps in the diagram.

Example 2.22. Continuing with our previous loopspace example, Let $\mathcal{X}$ be the punctured 3-cube where X_{123} is the based space X with basepoint $*$ and $X_S = *$ for all $S \in \mathcal{P}_0(\underline{3})$ such that $S \neq 123$. Then $\operatorname{holim}_{\mathcal{P}_0(\underline{3})}(\mathcal{X}) =$

$$\operatorname{holim}\left(\begin{tikzcd} & * \arrow[dl] \arrow[dr] & \\ * \arrow[r] & X & * \arrow[l] \\ * \arrow[u] \arrow[r] & * \arrow[u] & * \arrow[u] \arrow[l] \end{tikzcd}\right).$$

Since a point in $\operatorname{holim}(S_0)$ is a map $\Delta^2 \longrightarrow X$ such that $\partial\Delta^2$ is mapped to the basepoint $*$ of X, it follows that the homotopy limit is $\Omega^2 X \cong \operatorname{Map}(S^2, X)$.

Lemma 2.23. Given an n-cube $\mathcal{X}$ such that $n \geq 2$, there is a homeomorphism

$$\operatorname{holim}_{S \in \mathcal{P}_0(\underline{n})} X_S \cong \operatorname{holim}\Big(X_n \to \operatorname{holim}_{R \in \mathcal{P}_0(\underline{n-1})} X_{R \cup \{n\}} \leftarrow \operatorname{holim}_{R \in \mathcal{P}_0(\underline{n-1})} X_R\Big).$$

Remark 2.24. Note that the choice of X_n was arbitrary and was made for the sake of notational simplicity, i.e. the lemma could have been written in terms of any X_i.

Proof. The proof of Lemma 2.23 can be found in [MV, Lemma 5.3.6]. We illustrate the general idea of the proof below in the case where $n = 3$.

Let $\mathcal{X}$ be a punctured 3-cube. We want to show that there is a homeomorphism

$$\mathrm{holim}\left(\begin{tikzcd} & X_1 \arrow[dl] \arrow[dr] & \\ X_{13} \arrow[r] & X_{123} & X_{12} \arrow[l] \\ X_3 \arrow[u] \arrow[r] & X_{23} \arrow[u] & X_2 \arrow[u] \arrow[l] \end{tikzcd}\right) \cong \mathrm{holim}\left(\begin{tikzcd} X_1 \arrow[d, "f"] \\ \mathrm{holim}(X_{13} \to X_{123} \leftarrow X_{12}) \\ \mathrm{holim}(X_3 \to X_{23} \leftarrow X_2) \arrow[u, "g"'] \end{tikzcd}\right).$$

Let $\mathrm{holim}(S_1)$ be the homotopy limit on the right. To see that there is a homeomorphism, consider a point $(x_1, \gamma, (x_3, \alpha_{23}, x_2)) \in \mathrm{holim}(S_1)$. The map f sends x_1 to the point in $\mathrm{holim}(X_{13} \to X_{123} \leftarrow X_{12})$ which consists of a point $x_1|_{X_{12}}$ in X_{12}, a point $x_1|_{X_{13}}$ in X_{13} and the constant path $c|_{X_{123}}$ in X_{123} in a manner compatible with the maps in $\mathrm{holim}_{\mathcal{P}_0(\underline{3})}(\mathcal{X})$. The map g sends (x_3, α_{23}, x_2) to a point consisting of a point $x_3|_{X_{13}}$ in X_{13}, a point $x_2|_{X_{12}}$ in X_{12}, and a path $\alpha_{23}|_{X_{123}}$ in X_{123} once again in a manner compatible with $\mathrm{holim}_{\mathcal{P}_0(\underline{3})}(\mathcal{X})$.

Then the path γ between $f(x_1)$ and $g(x_3, \alpha_{23}, x_2)$ determines a one parameter path α_{13} between $x_1|_{X_{13}}$ and $x_3|_{X_{13}}$, a one parameter path between $x_1|_{X_{12}}$ and $x_2|_{X_{12}}$, and a two parameter path α_{123} between $c|_{X_{123}}$ and $\alpha_{23}|_{X_{123}}$ such that all the constructions are compatible. This determines a point in $\mathrm{holim}_{\mathcal{P}_0(\underline{3})}(\mathcal{X})$ and more generally, a map $F : \mathrm{holim}(S_1) \longrightarrow \mathrm{holim}_{\mathcal{P}_0(\underline{3})}(\mathcal{X})$ that sends each point in $\mathrm{holim}(S_1)$ to a unique point $(x_1, x_2, x_3, \alpha_{12}, \alpha_{13}, \alpha_{23}, \alpha_{123})$ in $\mathrm{holim}_{\mathcal{P}_0(\underline{3})}(\mathcal{X})$.

Similarly, we can show that there is map G from $\mathrm{holim}_{\mathcal{P}_0(\underline{3})}(\mathcal{X})$ to $\mathrm{holim}(S_1)$. It is easy to see that the compositions of these two maps are identity maps. Hence $\mathrm{holim}_{\mathcal{P}_0(\underline{3})}(\mathcal{X}) \cong \mathrm{holim}(S_1)$.

$\square$

Definition 2.25. Let $\mathcal{X}$ be an n-cube of based spaces. Then the **total fiber of** $\mathcal{X}$, denoted $\mathrm{tfiber}(\mathcal{X})$, is defined iteratively:

- If $\underline{n} = \emptyset$, then $\mathrm{tfiber}(\mathcal{X}) = X_\emptyset$.
- If $\underline{n} \neq \emptyset$, then consider the n-cube $\mathcal{X} = (\mathcal{Y} \to \mathcal{Z})$ as a map of $(n-1)$-cubes and define $\mathrm{tfiber}(\mathcal{X}) = \mathrm{hofiber}(\mathrm{tfiber}(\mathcal{Y}) \to \mathrm{tfiber}(\mathcal{Z}))$.

Remark 2.26. Since $\mathcal{X}$ is based, the total fiber of the $(n-1)-$cube $\mathcal{Z}$ has a natural basepoint given by the basepoints of the spaces in $\mathcal{Z}$ and constant homotopies.

Remark 2.27. It does not matter how $\mathcal{X}$ is considered as a map of cubes. For example, in the 2-cube, the total fiber is same whether we first take the homotopy fibers of $X_\emptyset \longrightarrow X_2$ and $X_1 \longrightarrow X_{12}$ and then find the total fiber or if we first take the homotopy fibers of $X_\emptyset \longrightarrow X_1$ and $X_2 \longrightarrow X_{12}$ and then find the total fiber. For more details, see [MV, Prop. 5.5.4].

Example 2.28. Let $\mathcal{X}$ be a 2-cube. Then the iterative process of finding the total fiber of $\mathcal{X}$ is illustrated below:

$$\begin{tikzcd}
\mathrm{tfiber}(\mathcal{X}) = \mathrm{hofiber}(f) \arrow[r] & \mathrm{hofiber}(X_\emptyset \longrightarrow X_2) \arrow[r, "f"] \arrow[d] & \mathrm{hofiber}(X_1 \longrightarrow X_{12}) \ . \arrow[d] \\
& X_\emptyset \arrow[r, "f_1"] \arrow[d, "f_2"'] & X_1 \arrow[d, "f_4"] \\
& X_2 \arrow[r, "f_3"] & X_{12}
\end{tikzcd}$$

Let x_0, x_1, x_2 and x_{12} be the basepoints of $X_\emptyset, X_1, X_2$, and X_{12} respectively. A point in $\mathrm{hofiber}_{x_2}(X_\emptyset \longrightarrow X_2)$ looks like (y, α) where $y \in X_\emptyset$ and α is a path from $f_2(y)$ to x_2. Then $f_1(y) \in X_1$ and $f_3(\alpha)$ is a path in X_{12} from $f_3(f_2(y))$ to the basepoint x_{12}. Define the map f as $(y, \alpha) \mapsto (f_1(y), f_3(\alpha))$. Note that $\mathrm{hofiber}_{x_{12}}(X_1 \longrightarrow X_{12})$ has basepoint $(x_1, c_{f_1(x_1)=x_{12}})$. Then a point in $\mathrm{tfiber}(\mathcal{X})$ consists of a point in $\mathrm{hofiber}_{x_2}(X_\emptyset \longrightarrow X_2)$ and a path of paths in $\mathrm{hofiber}_{x_{12}}(X_1 \longrightarrow X_{12})$. $\square$

If $\mathcal{X}$ is an n-cube in Top_* and S_0 is the punctured n-cube obtained by removing $X_\emptyset$ from $\mathcal{X}$, then there is a canonical map $a : X_\emptyset \longrightarrow \mathrm{holim}(S_0)$ that sends each point $x_0 \in X_\emptyset$ to the point in $\mathrm{holim}(S_0)$that consists of

- a point in each X_i, where $i \in \underline{n}$, obtained by mapping x_0 to X_i using the maps in the diagram
- constant homotopies elsewhere.

Example 2.29. Let $\mathcal{X}$ be the 2-cube

$$\begin{tikzcd}
X_\emptyset \arrow[r, "f"] \arrow[d, "g"'] & X_1 \arrow[d, "f'"] \\
X_2 \arrow[r, "g'"] & X_{12}.
\end{tikzcd}$$

Then we have the canonical map

$$\begin{aligned} a : X_\emptyset &\longrightarrow \mathrm{holim}(X_1 \longrightarrow X_{12} \longleftarrow X_2) \\ x &\longmapsto (f(x), c_{f'(f(x))=g'(g(x))}, g(x)) \end{aligned}$$

Proposition 2.30. Let $\mathcal{X}$ be an n-cube in Top_*. Let S_0 be the punctured n-cube resulting from removing $X_\emptyset$ from $\mathcal{X}$. Then $\mathrm{holim}(S_0) = \mathrm{Nat}(\Delta(\bullet), S_0)$ has a natural basepoint which sends each face $\Delta(S)$ of $\Delta(\underline{n})$ to the basepoint of the corresponding space X_S. Then

$$\mathrm{hofiber}(a : X_\emptyset \longrightarrow \mathrm{holim}(S_0)) \cong \mathrm{tfiber}(\mathcal{X})$$

Proof. The general proof of Proposition 2.30 can be found in [MV, Prop. 5.5.4]. We will illustrate the proof for the case $n = 2$. We proceed similarly to the proof of Lemma 2.23.

Let $\mathcal{X}$ be a 2-cube and S_0 the punctured n-cube resulting from removing $X_\emptyset$ from $\mathcal{X}$. We want to show that

$$\mathrm{tfiber}\left(\begin{tikzcd} X_\emptyset \arrow[r, "f"] \arrow[d, "g"'] & X_1 \arrow[d, "f'"] \\ X_2 \arrow[r, "g'"] & X_{12}. \end{tikzcd}\right) \cong \mathrm{hofiber}\left(X_\emptyset \longrightarrow \mathrm{holim}(S_0) \right).$$

Let x_0, x_1, x_2, x_{12} be the basepoints of X_0, X_1, X_2, X_{12} respectively. Note that the basepoint of $\mathrm{holim}(S_0) = \mathrm{holim}(X_1 \longrightarrow X_{12} \longleftarrow X_2)$ is $(x_1, c_{x_{12}}, x_2)$.

A point in $\mathrm{hofiber}\,(a : X_\emptyset \longrightarrow \mathrm{holim}(S_0))$ consists of a point $y \in X_\emptyset$ and a two-parameter path α from $(f(y), c_{f'(f(y))=g'(g(y))}, g(y))$ to the basepoint $(x_1, c_{x_{12}}, x_2)$. Then α determines the following:

- a path β from $(f(y), c_{f'(f(y))=g'(g(y))})$ to $(x_1, c_{x_{12}})$
- a path γ from $g(y)$ to x_2.

Since a point in $\mathrm{tfiber}(\mathcal{X})$ consists of a point (x, δ) and a path from $(f(x), g'(\delta))$ to $(x_1, c_{x_{12}})$ where δ is a path from $f(x)$ to x_2, it follows that the point y and the paths β and γ determine a point in $\mathrm{tfiber}(\mathcal{X})$. Hence, we have constructed a map $F : \mathrm{hofiber}(a) \longrightarrow \mathrm{tfiber}(\mathcal{X})$ that sends each point in $\mathrm{hofiber}(a)$ to a unique point in $\mathrm{tfiber}(\mathcal{X})$.

Similarly, we can define a map G from $\mathrm{tfiber}(\mathcal{X})$ to $\mathrm{hofiber}(a)$. It is easy to see that the compositions of F and G are identity maps. Hence $\mathrm{tfiber}(\mathcal{X}) \cong \mathrm{hofiber}\,(a : X_\emptyset \longrightarrow \mathrm{holim}(S_0))$. $\square$

Proposition 2.31. Let $\mathcal{X}$ be an n-cube. Append a point $*$ to $\mathcal{X}$ so that there is a unique map from $*$ to every X_S, $S \neq \emptyset$, and denote the augmented cube $\mathcal{Y}$. Then $\mathrm{holim}(\mathcal{Y}) \cong \mathrm{tfiber}(\mathcal{X})$.

Proof. We will work with the case where $n = 2$ for notational simplicity. However, the following proof can be easily modified for any n.

Let $\mathcal{Y}$ be a 2-cube $\mathcal{X}$ with a point appended in the manner described above. Then $\mathrm{holim}(\mathcal{Y}) =$

$$\mathrm{holim}\left(\begin{tikzcd} X_\emptyset \arrow[r] \arrow[d] & X_1 \arrow[d] & \\ X_2 \arrow[r] & X_{12} & \\ & & * \arrow[ul] \arrow[ull, bend left] \arrow[uul, bend right] \end{tikzcd}\right) = \mathrm{holim}\left(\begin{tikzcd} & X_\emptyset \arrow[dl] \arrow[d] \arrow[dr] & \\ X_1 \arrow[r] & X_{12} & X_2 \arrow[l] \\ & * \arrow[ul] \arrow[u] \arrow[ur] & \end{tikzcd}\right) \cong {}^{1}\mathrm{holim}\left(\begin{tikzcd} & X_\emptyset \arrow[dl] \arrow[d] \arrow[dr] & \\ X_1 \arrow[r] & X_{12} & X_2 \arrow[l] \\ * \arrow[u] \arrow[r, "Id"] & * \arrow[u] & * \arrow[u] \arrow[l, "Id"'] \end{tikzcd}\right).$$

[1]In order to prove this equivalence rigorously, we would need to define the homotopy limit for any diagram of spaces. This is beyond the scope of this paper. Intuitively, it is not hard to see that expanding the diagram at a point with the addition of identity maps and the same point does not change the homotopy limit.

By Lemma 2.23 and Example 2.20, it follows that

$$\operatorname{holim}\left(\begin{tikzcd} & & X_\emptyset \arrow[dl] \arrow[d] \arrow[dr] & & \\ & X_1 \arrow[r] & X_{12} & X_2 \arrow[l] & \\ * \arrow[ur] \arrow[rr, "Id"] & & * \arrow[u] & & * \arrow[ul] \arrow[ll, "Id"'] \end{tikzcd}\right) \cong \operatorname{holim}\left(\begin{tikzcd} X_\emptyset \arrow[d] \\ \operatorname{holim}(X_1 \to X_{12} \leftarrow X_2) \\ \operatorname{holim}(* \xrightarrow{Id} * \xleftarrow{Id} *) \arrow[u] \end{tikzcd}\right) \cong$$

$$\operatorname{holim}\left(\begin{tikzcd} X_\emptyset \arrow[d] \\ \operatorname{holim}(X_1 \to X_{12} \leftarrow X_2) \\ * \arrow[u] \end{tikzcd}\right) \cong \operatorname{hofiber}\Big(X_\emptyset \longrightarrow \operatorname{holim}(X_1 \longrightarrow X_{12} \longleftarrow X_2) \Big).$$

Then by Proposition 2.30

$$\operatorname{hofiber}\Big(X_\emptyset \longrightarrow \operatorname{holim}(X_1 \longrightarrow X_{12} \longleftarrow X_2) \Big) \cong \operatorname{tfiber}\left(\begin{tikzcd} X_\emptyset \arrow[r] \arrow[d] & X_1 \arrow[d] \\ X_2 \arrow[r] & X_{12} \end{tikzcd}\right).$$

□

Remark 2.32. In Section 6, we will freely use the fact that the homotopy limit satisfies a universal property in the sense if a space X maps to every space in a punctured n-cubical diagram $\mathcal{X}$, then there exists a unique map $X \longrightarrow \operatorname{holim}(\mathcal{X})$ that is compatible with the maps in $\mathcal{X}$. To prove that the homotopy limit satisfies this universal property rigorously requires a different perspective on homotopy limits that is beyond the scope of this paper. For details, see [MV, Chapter 8].

3 Khovanov Homology Interpreted via Cubical Diagrams

In [EvT], Everitt and Turner give a homotopy thereotic intepretation of Khovanov homology by associating the Khovanov homology of a knot diagram D with the homotopy limit of a diagram of Eilenberg-Mac Lane spaces. This is achieved by applying the Eilenberg-Mac Lane functor $K(-, n)$ to a commutative diagram of abelian groups obtained from D which yields an n-cube of spaces, whose homotopy limit has homotopy groups which are isomorphic to the unnormalised Khovanov homology of D.

The construction of Khovanov Homology as well as a brief sketch of Everitt and Turner's homotopy theoretic interpretation is shown below. We will adopt Everitt and Turner's notation in [EvT].

Finally, we will need to define a knot projection. A **knot projection** K is a projection of a knot onto a plane such that all the points of singularities are double points.

3.1 Khovanov Homology

Given an oriented knot projection D with crossings arbitrarily labeled $1, 2, ..., n$, let $\underline{n} = \{1, 2, ..., n\}$ be the set of crossings of D. Let $\mathbf{B}$ be the poset of subsets of $\underline{n}$ ordered by reverse inclusion, i.e. for $x, y \in \mathbf{B}$ such that $x \subseteq y$, we have $x \geq y$. By considering $\mathbf{B}$ as a category, we can construct a contravariant functor $F_{KH} : \mathbf{B}^{op} \longrightarrow \mathbf{Ab}$ that sends the n-cube $\mathbf{B}^{op}$ to a commutative diagram of abelian groups as follows:

Each crossing in D can be 0-resolved or 1-resolved as shown below:

For each $x \in \mathbf{B}^{op}$, a complete resolution $D(x)$ of D is a collection of planar circles which results from 1-resolving every crossing in x and 0-resolving every crossing not in x. For an example, see Figure 1.

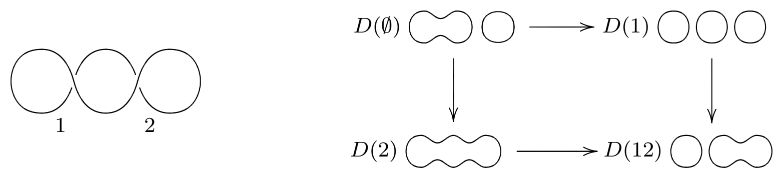


Figure 1: a projection of the unknot (left) and the corresponding diagram of complete resolutions (right)

Let V be the free abelian group generated by $\{1, s\}$. Let $F_{KH}(x) = V^{\otimes k}$ where k is the number of connected components of $D(x)$. For each $x, y \in \mathbf{B}^{op}$ where x is obtained by adding an element of $\underline{n}$ to y, denoted $x \prec y$, the complete resolution $D(x)$ results from 1-resolving a crossing that was 0-resolved in y. This amounts to two circles of $D(y)$ either merging into one circle or one circle of $D(y)$ splitting into two circles. Let $m : V \otimes V \longrightarrow V$ and $\Delta : V \longrightarrow V \otimes V$ be maps defined on generators as follows:

$$m : 1 \otimes 1 \longrightarrow 1,\ 1 \otimes s \longrightarrow s,\ s \otimes 1 \longrightarrow s,\ s \otimes s \longrightarrow 0$$
$$\Delta : 1 \longrightarrow 1 \otimes s + s \otimes 1,\ s \longrightarrow s \otimes s$$

If two circles merge, then $F_{KH}(x \prec y) : F_{KH}(y) = V^{\otimes k} \longrightarrow V^{\otimes k-1} = F_{KH}(x)$ defined by using m on the tensor factors corresponding to the merging circles and the identity on the rest. If one circle splits into two, then $F_{KH}(x \prec y) : F_{KH}(y) = V^{\otimes k} \longrightarrow V^{\otimes k+1} = F_{KH}(y)$ defined by using Δ on the tensor factor corresponding to the spliting circle and the identity on the rest. For an example, see Figure 2.

With this, the Khovanov cochain complex K^* can be constructed. Let $K^n = \oplus_{|x|=n} F_{KH}(x)$ be the n-cochain. In order to define the differentials, it suffices to make each face of the cube of abelian groups anticommute. This is achieved by sprinkling signs on the edges so that

$$\begin{tikzcd}
\emptyset & \{1\} \arrow[l] & & & V^{\otimes 2} \arrow[r, "\Delta"] \arrow[dd, "m"'] & V^{\otimes 3} \arrow[dd, "m"] \\
& & {} \arrow[r, "F_{KH}"] & {} & & \\
\{2\} \arrow[uu] & \{1,2\} \arrow[l] \arrow[uu] & & & V \arrow[r, "\Delta"] & V^{\otimes 2}
\end{tikzcd}$$

Figure 2: The functor $F_{KH} : \mathbf{B}^{op} \longrightarrow \mathbf{Ab}$ determined by the projection of the unknot in Figure 1.

each face contains an odd number of minus signs. Let $[x, y]$ denote the sign corresponding to the edge $F_{KH}(x \prec y)$. Let i denote the crossing that was 0-resolved in $D(y)$ but is 1-resolved in $D(x)$. Define $[x, y] = (-1)^{\sum_{j<i} 1}$ where the sum is over $j \in x$ and let $d : K^{n-1} \longrightarrow K^n$ be defined by $\sum [x, y] F_{KH}(x \prec y)$ such that $|x| = n$. For an example, see Figure 3.

$$\begin{tikzcd}
V^{\otimes 2} \arrow[r, "\Delta"] \arrow[d, "m"'] & V^{\otimes 3} \arrow[d, "m"] \\
V \arrow[r, "-\Delta"] & V^{\otimes 2}
\end{tikzcd}$$

Figure 3: a 2-cube that anticommutes due to a sprinkling of signs

Definition 3.1. The **unnormalised Khovanov homology** of a knot diagram D is the cohomology of the Khovanov cochain complex, i.e.

$$\overline{KH}^*(D) = H(K^*, d)$$

The unnormalised Khovanov homology is not invariant under isotopy. However, by normalising, a knot invariant is obtained:

Given an oriented knot diagram D, a crossing is said to be a negative crossing if the understrand runs from left to right. Otherwise, it is a positive crossing.

Figure 4: a negative crossing (left) and a positive crossing (right)

Definition 3.2. Let D be an oriented knot diagram. Let c be the number of negative crossings in D. Then the **normalised Khovanov homology** of D is defined as

$$KH^*(D) = \overline{KH}^{*+c}(D).$$

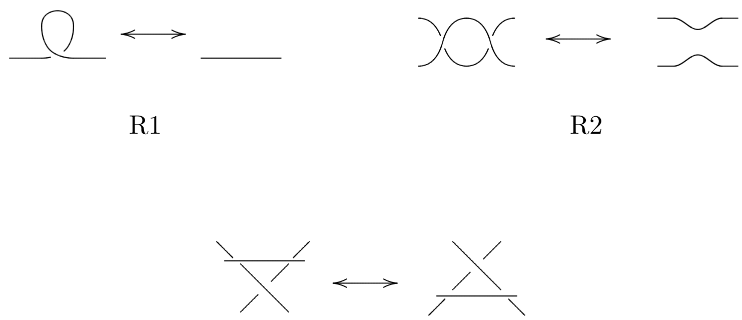


Figure 5: the three Reidemeister moves

We want to show that the normalised Khovanov homology is an invariant. It suffices to prove that Khovanov homology remains invariant under the three Reidemeister moves illustrated in Figure 5.

We will provide the proof for the Reidemeister I and Reidemeister II cases adapted from [BN]. For the proof of the Reidemeister III case, see [BN, 3.5.5]. We will need the following lemma.

Lemma 3.3. Let $\mathcal{C}$ be a chain complex and let $\mathcal{C}' \subset \mathcal{C}$ be a subchain complex. Then

- if $\mathcal{C}'$ is acyclic, i.e. its homology groups are zero, then $H(\mathcal{C}) \cong H(\mathcal{C}/\mathcal{C}')$ and
- if $\mathcal{C}/\mathcal{C}'$ is acyclic, then $H(\mathcal{C}) = H(\mathcal{C}')$.

Proof. Consider the long exact homology sequence

$$\dots \longrightarrow H^r(\mathcal{C}') \longrightarrow H^r(\mathcal{C}) \longrightarrow H^r(\mathcal{C}/\mathcal{C}') \longrightarrow H^{r+1}(\mathcal{C}') \longrightarrow \dots$$

associated with the sequence $0 \longrightarrow \mathcal{C}' \longrightarrow \mathcal{C} \longrightarrow \mathcal{C}/\mathcal{C}' \longrightarrow 0$. Suppose $\mathcal{C}'$ is acyclic. Then $H^r(\mathcal{C}') = 0$ for all r. So then the long exact homology sequence becomes

$$\dots \longrightarrow 0 \longrightarrow H^r(\mathcal{C}) \longrightarrow H^r(\mathcal{C}/\mathcal{C}') \longrightarrow 0 \longrightarrow \dots$$

By exactness, it follows that $H^r(\mathcal{C}) \cong H^r(\mathcal{C}/\mathcal{C}')$. The proof for the second part is similar.

Now suppose that $\mathcal{C}/\mathcal{C}'$ is acyclic. Then $\mathcal{H}^r(\mathcal{C}/\mathcal{C}') = 0$ for all r. So then the long exact homology sequence becomes

$$\dots \longrightarrow 0 \longrightarrow H^r(\mathcal{C}') \longrightarrow H^r(\mathcal{C}) \longrightarrow 0 \longrightarrow \dots$$

Then by exactness, it follows that $H(\mathcal{C}) = H(\mathcal{C}')$. $\square$

Proposition 3.4. The normalised Khovanov homology is invariant under the first Reidemeister move R1.

Proof. Let K be a knot projection and K' be K with a twist in the projection. We want to show that $\mathrm{KH}^r(K) \cong \mathrm{KH}^r(K')$ for all r. For the sake of notational simplicity, let $\mathcal{H}^r(K) = \mathrm{KH}^r(K)$ and $\mathcal{H}^r(K') = \mathrm{KH}^r(K')$.

Let $[\![\ ]\!]$ denote the cochain complex generated by K'. Note that the n-cube generated by K' can be considered as a map between two $(n-1)$-cubes where one $(n-1)$-cube $\mathcal{X}$ consists of all the abelian groups corresponding to complete resolutions where the twist is 0-resolved and the other $(n-1)$-cube $\mathcal{Y}$ consists of all abelian groups corresponding to complete resolutions where the twist is 1-resolved. Then let $[\![\ ]\!]$ denote the subcomplex generated by $\mathcal{X}$ and $[\![\ ]\!]$ denote the subcomplex generated by $\mathcal{Y}$.

Then we have the complex

$$\mathcal{C} = [\![\ ]\!] = [\![\ ]\!] \xrightarrow{m} [\![\ ]\!].$$

We want to consider the following subcomplex of $\mathcal{C}$.

$$\mathcal{C}' = [\![\ ]\!]_1 \xrightarrow{m} [\![\ ]\!].$$

We need to explain the notation above. In particular, we want to construct the subcomplex $[\![\ ]\!]_1$ of $[\![\ ]\!]$. Each cochain of $[\![\ ]\!]$ consists of linear combinations of tensor products that correspond to the cycles in the complete resolutions. Each of these complete resolutions has the cycle from 0-resolving the twist. Then define $[\![\ ]\!]_1$ to be the subcomplex whose cochains consists of elements such that the component of the element corresponding to the cycle in is 1. Note that 1 is a unit for the map m. So then the restriction of m to $[\![\ ]\!]_1$ is an isomorphism and so $\mathcal{C}'$ is acyclic. Hence by Lemma 3.3, it follows that $\mathcal{H}(\mathcal{C}) \cong \mathcal{H}(\mathcal{C}/\mathcal{C}')$.

Now consider the quotient complex

$$\begin{aligned}\mathcal{H}(\mathcal{C}/\mathcal{C}') &= \mathcal{H}([\![\ ]\!]/[\![\ ]\!]_1 \xrightarrow{m} 0) \\ &\cong \mathcal{H}([\![\ ]\!]/[\![\ ]\!]_1).\end{aligned}$$

Since we have essentially "mod out by $1 = 0$" from the tensor factor V corresponding to the cycle in the twist, it follows that any element of a cochain of $[\![\ ]\!]/[\![\ ]\!]_1$ has s for the component of the element corresponding to V. So then $V/(1 = 0)$ is generated by a single element s. But this means that $V/(1 = 0) \otimes V^{\otimes k} \cong V^{\otimes k}$. So $\mathcal{H}(K') = \mathcal{H}([\![\ ]\!]) \cong \mathcal{H}([\![\ ]\!]/[\![\ ]\!]_1) \cong \mathcal{H}([\![\ ]\!]) = \mathcal{H}(K)$. Thus, the normalised Khovanov homology is invariant under R1.

□

Proposition 3.5. The normalised Khovanov homology is invariant under the second Reidemeister move R2.

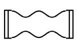
Proof. The proof of invariance under R2 uses the same techniques as the ones in the proof of Proposition 3.4. We will provide a diagrammatic sketch of the proof similar to [BN, Figure 1] with the understanding that the details can be filled easily.

For the sake of notational simplicity, let $[\![K]\!]_{/1=0} := [\![K]\!]/[\![K]\!]_1$ and

$A = [\![$  $]\!]$

$B = [\![$  $]\!]$

$C = [\![$  $]\!]$

$D = [\![$  $]\!]$.

Then we have the following cochain complex:

$\mathcal{C} = [\![$  $]\!]$ $=$

$$\begin{tikzcd} A \arrow[r, "m"] & B \\ C \arrow[u, "\Delta"] \arrow[r] & D \arrow[u] \end{tikzcd}$$

Consider the subcomplex

$$\mathcal{C}' = \begin{tikzcd} A_1 \arrow[r, "m"] & B \\ 0 \arrow[u, "\Delta"] \arrow[r] & 0 \arrow[u] \end{tikzcd}$$

Using the same argument as Proposition 3.4, it follows that $\mathcal{C}'$ is acyclic. Then we have

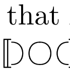
$$\mathcal{C}' = \begin{tikzcd} A_{/1=0} \arrow[r, "m"] & 0 \\ C \arrow[u, "\Delta"] \arrow[r] & D \arrow[u] \end{tikzcd}$$

Now consider the subcomplex $\mathcal{C}''$ of $\mathcal{C}/\mathcal{C}'$ illustrated below:

$$\mathcal{C}'' = \begin{tikzcd} 0 \arrow[r] & 0 \\ 0 \arrow[u] \arrow[r] & D \arrow[u] \end{tikzcd}$$

Consider the quotient complex

$$(\mathcal{C}/\mathcal{C}')/\mathcal{C}'' = \begin{tikzcd} A_{/1=0} \arrow[r] & 0 \\ C \arrow[u, "\Delta"] \arrow[r] & 0 \arrow[u] \end{tikzcd}$$

We want to prove that Δ is an isomorphism. Clearly Δ is an injection. Consider an element x of a cochain of $[\![$  $]\!]$. Then the components of x corresponding to the middle and right cycles of $[\![\supset\bigcirc\subset]\!]$ is either 1 and s or 1 and 1. In the first case, the element y in $[\![\supset\asymp]\!]$ with all components equal outside of the R2 region with s corresponding to the cycle on the right in $[\![\supset\asymp]\!]$ maps to x under Δ. Similarly for the second case. So then Δ is surjective and thus an isomorphism. This means that $(\mathcal{C}/\mathcal{C}')/\mathcal{C}''$ is acyclic. Then by Lemma 3.3, it follows that $\mathcal{H}(\mathcal{C}) \cong \mathcal{H}(\mathcal{C}/\mathcal{C}') = \mathcal{H}(\mathcal{C}'')$. Thus we have invariance under R2. $\square$

Example 3.6. We will show an example of the construction of the Khovanov cochain complex as well as a computation of Khovanov homology.

Consider the projection D from Figure 1. It generates the following 2-cube

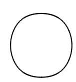
$$\begin{tikzcd}
V^{\otimes 2} \arrow[r, "\Delta"] \arrow[d, "m"'] & V^{\otimes 3} \arrow[d, "m"] \\
V \arrow[r, "-\Delta"] & V^{\otimes 2}
\end{tikzcd}$$

.

This yields the following cochain complex:

$$\ldots \longrightarrow 0 \longrightarrow V^{\otimes 2} \xrightarrow{\Delta \oplus m} V^{\otimes 3} \oplus V \xrightarrow{(m,-\Delta)} V^{\otimes 2} \longrightarrow 0 \longrightarrow \ldots$$

Since the projection D is the unknot, it suffices to calculate Khovanov homology for the projection D'



We have the following cochain complex:

$$\ldots \xrightarrow{d_{-2}} 0 \xrightarrow{d_{-1}} V \xrightarrow{d_0} 0 \xrightarrow{d_1} \ldots$$

Then

$$\mathrm{KH}^*(D') = \begin{cases} \mathrm{KH}^0(D') = V/0 = V \\ \mathrm{KH}^r(D') = 0, & \text{for all } r \neq 0 \end{cases}$$

3.2 A Homotopy Theoretic Interpretation of Khovanov Homology

In order to arrive at Everitt and Brent's homotopy theoretic interpretation of Khovanov homology, the n-cube $\mathbf{B}^{op}$ will have to be modified in a manner that does not affect the construction of the unnormalised Khovanov Homology. This is because $\mathbf{B}^{op}$ has an initial object and as a result, any n-cube of spaces obtained from $\mathbf{B}^{op}$ will have a homotopy limit that is equivalent to the initial space.

We modify $\mathbf{B}$ by appending a point $*$ so that for any $x \neq \emptyset$, there is a unique morphism from x to $*$ and denote the augmented diagram $\mathbf{Q}$. Then the functor $F_{KH} : \mathbf{B}^{op} \longrightarrow \mathbf{Ab}$ can be extended to $F_{KH} : \mathbf{Q}^{op} \longrightarrow \mathbf{Ab}$ by letting $F_{KH}(*) = 0$ and $F_{KH}(* \to x) : F_{KH}(*) \longrightarrow F_{KH}(x)$ be the only possible homomorphism. The cochain complex over $\mathbf{Q}^{op}$

is constructed in the same manner as section 4.1 with minor additions. Let $[x,*] = -1$ for each $x \prec *$ and define the signs of all other edges $[x,y]$ as before. The direct sum $K^0 = \oplus_{|x|=0} F_{KH}(x)$ includes the zero group and the corresponding differential sums over the added homomorphisms. Then the resulting cohomology is precisely the unnormalised Khovanov homology. Thus, adding a point to the diagram in the manner described above preserves Khovanov homology.

Definition 3.7. An **Eilenberg-Mac Lane space** $K(G,n)$ is a space with only one non-trivial homotopy group $\pi_n(X) \cong G$.

We can apply the Eilenberg-Mac Lane functor $K(-,n)$ to a diagram of abelian groups. It sends each group G to the Eilenberg-Mac Lane space $K(G,n)$ and homomorphisms between the groups in the diagram to maps of spaces determined by the cell structures of the corresponding Eilenberg-Mac Lane spaces. By composing $K(-,n)$ with the functor F_{KH}, we obtain a diagram of Eilenberg-Mac Lane spaces. This is a cubical diagram of spaces with an additional point $K(0,n) \simeq *$ mapping to all spaces except the initial space. The following theorem associates Khovanov homology with the nth homotopy group of the homotopy limit of this diagram of spaces.

Theorem 3.8. (Everitt and Turner) Let $\mathbf{Y}_n D = \mathrm{holim}_{\mathbf{Q}^{op}}(K(-,n) \circ F_{KH})$. Then

$$\pi_i(\mathbf{Y}_n D) \cong \begin{cases} \overline{KH}^i(D) & 0 \le i \le n \\ 0 & \text{else.} \end{cases}$$

Proposition 3.9.

$$\mathbf{Y}_n D = \mathrm{tfiber}_{\mathbf{B}^{op}}(K(-,n) \circ F_{KH})$$

Proof. This follows immediately from Proposition 2.31 since the homotopy limit of a cubical diagram augmented by a point is the total fiber of the cubical diagram. □

4 Calculus of Functors

The idea of calculus of functors is that the behavior of a functor $F : \mathcal{C} \longrightarrow \mathcal{C}'$ can be approximated by a sequence of simpler functors $T_k F : \mathcal{C} \longrightarrow \mathcal{C}'$. The functors $T_k F$ are constructed so that we have natural transformations $F \longrightarrow T_k F$ compatible with the composition $F \longrightarrow T_{k+1} F \longrightarrow T_k F$. This results in a commutative diagram illustrated below:

$$
\begin{tikzcd}
 & \vdots \arrow[d] \\
 & T_{k+1}F \arrow[d] \\
F \arrow[ur] \arrow[r] \arrow[dr] \arrow[dddr] & T_kF \arrow[d] \\
 & T_{k-1}F \arrow[d] \\
 & \vdots \arrow[d] \\
 & T_0F
\end{tikzcd}
$$

This is called a **Taylor tower**. Each T_kF is called the k-th stage of the Taylor tower.

The construction of a Taylor tower for spaces of long knots lies in the domain of what is called **manifold calculus of functors**. Let M be a smooth, closed manifold of dimension m. Let $\mathcal{O}(M)$ be the category that has the open sets of M as objects and inclusions as morphisms. Manifold of calculus studies contravariant functors $F : \mathcal{O}(M) \longrightarrow \mathrm{Top}$. Of particular interest to us is the space of embeddings $\mathrm{Emb}(M, N)$ where N is a smooth manifold, which can be regarded as a contravariant functor.

Definition 4.1. Let M and N be smooth manifolds. Then

- an **immersion** is a smooth map f such that its derivative $df : TM \longrightarrow TN$ is a fiberwise injection and
- an **embedding** is a immersion f with the additional condition that it is a homeomorphism onto its image.

We denote the **space of immersions** $\mathrm{Imm}(M, N)$ and the **space of embeddings** $\mathrm{Emb}(M, N)$. They are topologized with the Whitney C^∞ topology. For details on this topology, see [MV, Appendix A]. Since every embedding is an immersion, it immediately follows that $\mathrm{Emb}(M, N) \subseteq \mathrm{Imm}(M, N)$.

To see how the space of embeddings is a functor $\mathrm{Emb}(-, N)$, consider two open sets $U_1, U_2 \subseteq M$ such that $U_1 \subseteq U_2$. The inclusion $U_1 \hookrightarrow U_2$ corresponds to the restriction $\mathrm{Emb}(U_2) \longrightarrow \mathrm{Emb}(U_1)$.

In the following sections we will restrict our attention to the case where $M = \mathbb{R}$ and $N = \mathbb{R}^3$. That is, we will build a Taylor tower specifically for the space of long knots rather than a general construction for a space of embeddings.

4.1 Taylor tower for spaces of long knots

Since knots are embeddings, it seems natural to study them in the context of manifold calculus. The Taylor tower for knots consists of the space of knots for F and homotopy limits of punctured diagrams for each T_kF. Traditionally, knots are defined as embeddings of S^1 in $\mathbb{R}^3$. However, it will be more convenient for us to consider knots as embeddings of $\mathbb{R}$ in $\mathbb{R}^3$.

Definition 4.2. Let $e : \mathbb{R} \longrightarrow \mathbb{R}^3$ be the standard linear embedding defined by $t \longmapsto (t, 0, 0)$. Then the **space of long knots** $\mathrm{Emb}_c(\mathbb{R}, \mathbb{R}^3)$ consists of embeddings f such that f agrees with e on all points outside of a compact set, i.e.

$$\mathrm{Emb}_c(\mathbb{R}, \mathbb{R}^3) = \{f \in \mathrm{Emb}(\mathbb{R}, \mathbb{R}^3) : f(t) = (t, 0, 0) \text{ for all } t \notin [0, 1]\}.$$

It is topologized as a subspace of $\mathrm{Emb}(\mathbb{R}, \mathbb{R}^3)$. The space of immersions $\mathrm{Imm}_c(\mathbb{R}, \mathbb{R}^3)$ is defined similarly.

Intuitively, it is not hard to see how a closed knot in $\mathrm{Emb}(S^1, \mathbb{R}^3)$ relates to a long knot in $\mathrm{Emb}(\mathbb{R}, \mathbb{R}^3)$. A long knot simply restricts all the "tangling" of the knot to a compact set. Outside of this compact set, a long knot is a straight line that approaches $\pm\infty$. We can think of this as the knot closing up at infinity. A more rigorous treatment of how closed knots and long knots are related can be found in [MV, 10.3.1].

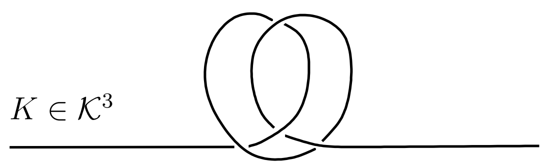
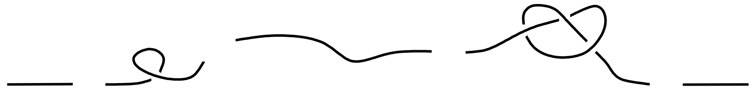


Figure 6: A long knot $K \in \mathcal{K}^3$

Note that paths in $\mathrm{Emb}_c(\mathbb{R}, \mathbb{R}^3)$ are isotopies. The homotopy group $\pi_0(\mathrm{Emb}(\mathbb{R}, \mathbb{R}^3))$ is the set of all isotopy classes of knots and the cohomology $H^0(\mathrm{Emb}(\mathbb{R}, \mathbb{R}^3))$ is the space of all knot invariants which are functions that give the same value for isotopic knots.

We will now construct the Taylor tower for long knots. In particular, we want to define the stages of the Taylor tower.

Definition 4.3. Let $\mathcal{K}^3$ denote the space of long knots $\mathrm{Emb}(\mathbb{R}, \mathbb{R}^3)$. Let $I_1, \ldots, I_{k+1}$ be disjoint subintervals of $\mathbb{R}$ and S a nonempty subset of $\underline{k+1} = \{1, 2, \ldots, k+1\}$. Then define the space $\mathcal{K}^3_S = \mathrm{Emb}(\mathbb{R} \setminus \bigcup_{i \in S} I_i, \mathbb{R}^3)$.



an element of $\mathcal{K}^3_{\{1,2,3,4\}}$

Let $i \in \{1, 2, \ldots, k+1\} \backslash S$. Then there is a map $\mathcal{K}^3_S \longrightarrow \mathcal{K}^3_{S \cup \{i\}}$ defined by restrictions of embeddings to embeddings with one more puncture. These restrictions and the spaces $\mathcal{K}^3_S$ define a punctured $(k+1)$-cubical diagram. Let $X_k\mathcal{K}^3$ denote this punctured $(k+1)$-cube.

Example 4.4. When $k = 2$, we have the following punctured 3-cube:

$$\begin{tikzcd}
 & & \mathcal{K}^3_{\{1\}} \arrow[dd] \arrow[dr] & \\
 & \mathcal{K}^3_{\{2\}} \arrow[rr] \arrow[dd] & & \mathcal{K}^3_{\{1,2\}} \arrow[dd] \\
\mathcal{K}^3_{\{3\}} \arrow[rr] \arrow[dr] & & \mathcal{K}^3_{\{1,3\}} \arrow[dr] & \\
 & \mathcal{K}^3_{\{2,3\}} \arrow[rr] & & \mathcal{K}^3_{\{1,2,3\}}.
\end{tikzcd}$$

Definition 4.5. The k-th stage of the Taylor tower for $\mathcal{K}^3$ is the homotopy limit of the punctured $(k+1)$-cube defined by $\mathcal{K}^3_S$, i.e.

$$T_k\mathcal{K}^3 = \operatorname{holim}(X_k\mathcal{K}^3)$$

Example 4.6. Consider, as in Example 4.4, the case where $k = 2$. Then we have the following punctured 3-cube:

$$\begin{tikzcd}
 & & \mathcal{K}^3_{\{1\}} \arrow[dl] \arrow[dr] & & \\
 & \mathcal{K}^3_{\{1,3\}} \arrow[r] & \mathcal{K}^3_{\{1,2,3\}} & \mathcal{K}^3_{\{1,2\}} \arrow[l] & \\
\mathcal{K}^3_{\{3\}} \arrow[ur] \arrow[rr] & & \mathcal{K}^3_{\{2,3\}} \arrow[u] & & \mathcal{K}^3_{\{2\}} \arrow[ll] \arrow[ul]
\end{tikzcd}$$

A point in $T_2\mathcal{K}^3$ consists of

- one-punctured knots in $\mathcal{K}^3_{\{1\}}$, $\mathcal{K}^3_{\{2\}}$, $\mathcal{K}^3_{\{3\}}$
- isotopies of twice-punctured knots in $\mathcal{K}^3_{\{1,2\}}$, $\mathcal{K}^3_{\{1,3\}}$, $\mathcal{K}^3_{\{2,3\}}$
- a two-parameter isotopy of a thrice-punctured knot in $\mathcal{K}^3_{\{1,2,3\}}$

such that they are all compatible with the restriction maps in the diagram.

It remains to define the natural transformations in the Taylor tower. Since $\mathcal{K}^3$ maps to each space in the punctured cube $X_k\mathcal{K}^3$ via restriction maps (essentially punching holes into a knot), it follows that there is a unique map $\mathcal{K}^3 \longrightarrow T_k\mathcal{K}^3$ (isotopies are constant). Furthermore since the homotopy limit of a cubical diagram maps in a canonical way to every space in the diagram and every $X_k\mathcal{K}^3$ is a subdiagram contained in $X_{(k+1)}\mathcal{K}^3$, by Lemma 2.23, it follows that there is a canonical map $T_{(k+1)}\mathcal{K}^3 \longrightarrow T_k\mathcal{K}^3$. Combining these maps yields the Taylor tower for long knots:

$$
\begin{tikzcd}
 & \vdots \arrow[d] \\
 & T_{k+1}\mathcal{K}^3 \arrow[d] \\
\mathcal{K}^3 \arrow[ur] \arrow[r] \arrow[dr] \arrow[dddr] & T_k\mathcal{K}^3 \arrow[d] \\
 & T_{k-1}\mathcal{K}^3 \arrow[d] \\
 & \vdots \arrow[d] \\
 & T_0\mathcal{K}^3
\end{tikzcd}
$$

Now that we have constructed the Taylor tower for long knots, we want to know if this Taylor tower converges, i.e. is it true that $\pi_*(\mathcal{K}^3) \cong \pi_*(T_\infty\mathcal{K}^3)$, where $T_\infty\mathcal{K}^3$ is the inverse limit of the tower? Similarly, do we have $H^*(\mathcal{K}^3) \cong H^*(T_\infty\mathcal{K}^3)$? It is not known if the Taylor tower converges. However we can measure whether the map $\mathcal{K}^3 \longrightarrow T_k\mathcal{K}^3$ induces isormophisms on cohomology through a range by studying the difference between $\mathcal{K}$ and $T_k\mathcal{K}^3$, i.e. the homotopy fiber of the map $\mathcal{K}^3 \longrightarrow T_k\mathcal{K}^3$. Namely, for a fiber sequence $\mathrm{hofiber}(f) \longrightarrow X \xrightarrow{f} Y$, there is a long exact sequence

$$\ldots \longrightarrow \pi_n(\mathrm{hofiber}(f)) \longrightarrow \pi_n(X) \longrightarrow \pi_n(Y) \longrightarrow \pi_{n-1}(\mathrm{hofiber}(f)) \longrightarrow \ldots.$$

So knowing $\pi_*(\mathrm{hofiber}(f))$ (or $H^*(\mathrm{hofiber}(f))$) tells us how close X and Y are on homotopy. But by Proposition 2.30, the homotopy fiber of $\mathcal{K}^3 \longrightarrow T_k\mathcal{K}^3$ is also the total fiber of the $k+1$-cube $\mathcal{K} \longrightarrow \mathcal{X}_k\mathcal{K}^3$, i.e.

$$\mathrm{tfiber}(\mathcal{K} \longrightarrow \mathcal{X}_k\mathcal{K}^3) \;\cong\; \mathrm{hofiber}(\mathcal{K} \longrightarrow T_k\mathcal{K}^3)$$

4.2 Equivalent constructions of the Taylor tower for spaces of long knots

In practice, it is useful to construct the Taylor tower for spaces of long knots with what is called the "space of knots modulo immersions" rather than with $\mathrm{Emb}_c(\mathbb{R}, \mathbb{R}^3)$ and relate the new construction to configuration spaces. It is beyond the scope of this paper to explicitly show how these modifications are related to our original construction or that they do not significantly change the context in which our original construction studies knots. Instead, we provide some intuition for this new construction because it lends a valuable perspective in the study of knots.

The procedure for building the Taylor tower with the space of knots modulo immersions is essentially identical with the one in Section 4.1. We will denote the space of knots modulo immersions $\overline{\mathcal{K}}^3$. We will write $\mathrm{Emb}_c(\mathbb{R}, \mathbb{R}^3)$ when referring to the space of long knots defined in Section 4.1.

Definition 4.7. Let $\mathrm{Emb}_c(\mathbb{R}, \mathbb{R}^3) \hookrightarrow \mathrm{Imm}_c(\mathbb{R}, \mathbb{R}^3)$ be the inclusion map. Then the **space of knots modulo immersions** is defined as

$$\overline{\mathcal{K}}^3 = \mathrm{hofiber}_e(\mathrm{Emb}_c(\mathbb{R}, \mathbb{R}^3) \hookrightarrow \mathrm{Imm}_c(\mathbb{R}, \mathbb{R}^3)).$$

So a point in $\overline{\mathcal{K}}^3$ consists of a long knot K and an isotopy from K to the unknot e.

Definition 4.8. Let $I_1, \ldots, I_{k+1}$ be disjoint subintervals of $\mathbb{R}$ and S a nonempty subset of $\underline{k+1} = \{1, 2, \ldots, k+1\}$. Let e_S denote the restriction of e to $\mathbb{R} \backslash \bigcup_{i \in S}$. Then define

$$\overline{\mathcal{K}}^3_S = \mathrm{hofiber}_{e_S}(\mathrm{Emb}(\mathbb{R} \backslash \bigcup_{i \in S} I_i, \mathbb{R}^3) \hookrightarrow \mathrm{Imm}(\mathbb{R} \backslash \bigcup_{i \in S} I_i, \mathbb{R}^3)).$$

Let $i \in \{1, 2, \ldots, k+1\} \backslash S$. Then similar to the construction in Section 4.1, there is a map $\overline{\mathcal{K}}^3_S \longrightarrow \overline{\mathcal{K}}^3_{S \cup \{i\}}$ defined by restrictions of embeddings to embeddings with one more puncture. These restrictions and the spaces $\overline{\mathcal{K}}^3_S$ define a punctured $(k+1)$-cubical diagram. Let $X_k \overline{\mathcal{K}}$ denote this punctured $(k+1)$-cube.

Definition 4.9. The k-th stage of the Taylor tower for $\overline{\mathcal{K}}^3$ is the homotopy limit of the punctured $(k+1)$-cube defined by $\overline{\mathcal{K}}^3_S$, i.e.

$$T_k \overline{\mathcal{K}}^3 = \mathrm{holim}(X_k \overline{\mathcal{K}})$$

The natural transformations in this modified construction of the Taylor tower arise in same matter as described in Section 4.1. Hence, combining these maps with our newly defined k-th stages yields a Taylor tower for the space of knots modulo immersions.

We will now associate to each space $\overline{\mathcal{K}}^3_S$ a configuration space.

Definition 4.10. The **configuration space of n points in** $\mathbb{R}$ is defined as

$$\mathrm{Conf}(n, \mathbb{R}^3) = \{(x_1, x_2, \ldots, x_n) \in (\mathbb{R}^3)^n : x_i \neq x_j \text{ for all } i \neq j\}$$

Example 4.11. From the definition, it is easy to see that

- $\mathrm{Conf}(0, \mathbb{R}^3) = *$
- $\mathrm{Conf}(1, \mathbb{R}^3) = \mathbb{R}^3$.

The space $\overline{\mathcal{K}}^3_S$ contains knots with $|S|$ many punctures. These punctured knots consist of disjoint arcs. If we retract each arc to its midpoint, then we are left with $|S| - 1$ points. So $\overline{\mathcal{K}}^3_S \simeq \mathrm{Conf}(|S| - 1, \mathbb{R}^3)$. Then intuitively, the restriction maps $\overline{\mathcal{K}}^3_S \longrightarrow \overline{\mathcal{K}}^3_{S \cup \{i\}}$ correspond to the maps $\mathrm{Conf}(|S| - 1, \mathbb{R}^3) \longrightarrow \mathrm{Conf}(|S|, \mathbb{R}^3)$ that "add a point" as shown in Figure 7.

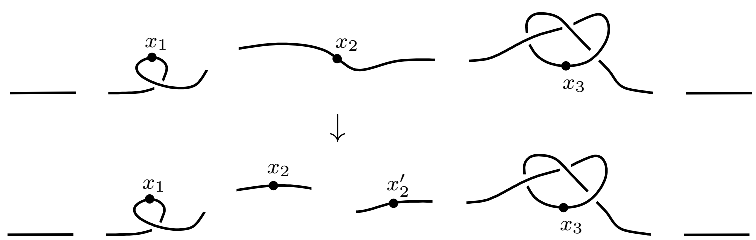


Figure 7: The restriction map "adds a point"

Thus each kth stage $T_k\overline{\mathcal{K}}^3$ is "built out of" configuration spaces, i.e. intuitively, we have

$$T_3\overline{\mathcal{K}}^3 \simeq \operatorname{holim}\left(\begin{tikzcd} & & \mathrm{Conf}(0,\mathbb{R}^3) \arrow[dr] \arrow[dd] & \\ & \mathrm{Conf}(0,\mathbb{R}^3) \arrow[rr] \arrow[dd] & & \mathrm{Conf}(1,\mathbb{R}^3) \arrow[dd] \\ \mathrm{Conf}(0,\mathbb{R}^3) \arrow[rr] \arrow[dr] & & \mathrm{Conf}(1,\mathbb{R}^3) \arrow[dr] & \\ & \mathrm{Conf}(1,\mathbb{R}^3) \arrow[rr] & & \mathrm{Conf}(2,\mathbb{R}^3). \end{tikzcd}\right)$$

We need cosimplicial spaces to make the maps between configuration spaces more precise. For details, see [MV, 10.3.2].

5 Conjectures

In [Vol], the Taylor tower for the spaces of long knots is shown to contain information about finite type invariants. In particular, the Jones polynomial is encoded in the invariants of the Taylor tower in the following manner: the coefficient of the nth degree term of the Jones polynomial is an element of $H^0(T_{2n}\mathcal{K}^3)$. Since the Jones polynomial can be extracted from Khovanov homology, it is natural to ask whether the higher homology of the Taylor tower contains information about Khovanov homology.

Furthermore, Everitt and Turner's space level realization of Khovanov homology is constructed using similar cubical diagram techniques to the ones used in manifold calculus of functors and the construction of the Taylor tower for knots. In particular, the total fiber of a cube of punctured knots can be used to study the difference between $\mathcal{K}^3$ and $T_k\mathcal{K}^3$ (See end of Section 4.1). This connection is made further explicit in Proposition 3.9. This leads to the following conjectures:

Conjecture 1. The cohomology of the stages of the Taylor tower for the spaces of long knots is related to Khovanov homology. More precisely, let K be a knot with crossing number n. Let $H^*(T_{2n}(K))$ denote the cohomology of the $2n$-th Taylor stage component of the space of knots containing the knot K. Then $H^*(T_{2n}(K)) \supset \mathrm{KH}^*(K)$.

Conjecture 2. $H^*(T_{2n}(K))$ is built out of $H^*(\mathrm{Conf}(k,\mathbb{R}^3))$. Thus $\mathrm{KH}^*(K)$ can be represented through the cohomology of configuration spaces.

References

[BN] Dror Bar-Natan *On Khovanov's categorification of the Jones polynomial*, Algebraic and Geometric Topology 2 (2002), 337-370.

[EvT] Brent Everitt and Paul Turner. *The homotopy theory of Khovanov Homology*, arXiv:math.GT/1112.3460v4.

[MV] Brian Munson and Ismar Volić. *Cubical homotopy theory*, work in progress.

[Vol] Ismar Volić. *Finite type knot invariants and calculus of functors*, Compositio Mathematica 142 (2006), 222-250.