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Index Sets of some Computable Groups

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Contents

1 Abstract

We say that e is a computable index of a computable group G if the partial computable function φ_e computes the atomic diagram of G. The set of all such indicies is the index set of G. The complexity of index sets is one measure of the computational complexity of the group. In [3] Carson et al. showed that the index set of free groups of rank n is d-Σ_2 complete. We expand on this result to show that the index sets of groups with presentation of the form
$\langle x_1, \dots x_m, x_{m+1}, \dots x_n | x_1^2 = \dots = x_m^2 = 1\rangle$ is also d-Σ_2 complete. Additionally, we investigate the complexity of index sets of certain torsion-free abelian groups of rank 1. In [10] Saraph showed that every torsion-free abelian group of rank 1 has index set d-Σ_2 or Σ_3 depending on the divisibility properties of the groups. We investigate remaining gaps in Saraph's characterization.

2 Introduction

This work is in the area of computable structure theory, which focuses on the complexity encoded in fixed mathematical objects. Computability theory is a branch of mathematical logic that makes rigorous the notion of being algorithmic. All of the sets we consider are subsets of the natural numbers, denoted ω. We say that a set is *computable* if there exists some algorithm that determines exactly which elements are in the set and which are not. A set is *computably enumerable* (c.e.) if there exists an algorithm that enumerates all of the elements of the set. This is a slightly weaker condition, and there are sets that are computably enumerable but not computable, e.g. the *halting set*. Computability provides us with tools to examine computational information encoded in a given object. We restrict our investigation to computable groups. Specifically, we examine generalizations of free groups. Two structures are *elementarily equivalent* if they satisfy the same first-order sentences. In 1945 Tarski asked whether two free groups of differing, but finite, rank are *elementarily equivalent*. 50 years later, Sela gave a positive answer to Tarski's question [18] [12] [14] [13] [15] [16] [17]. Thus, first-order sentences cannot be used to distinguish free groups of differing rank. However, such groups can be differentiated using more expressive logics. Carson et.al employed computable infinitary sentences to distinguish these free groups of differing ranks [3]. Here we expand this work to investigate the complexity of the descriptions of groups similar to the free groups.

We examine two kinds of infinite groups, specifically finitely presented groups and torsion-free abelian groups of rank 1. In [10] Saraph investigated the index set complexity for torsion-free abelian groups of rank 1. Saraph found that every torsion-free abelian group of rank 1 has index set complexity d-Σ_2 or Σ_3, and found the exact index set complexity for some torsion-free abelian groups of rank 1. We investigate remaining open cases in his characterization.

3 Preliminaries

3.1 Computable Structure Theory

We focus on the complexity of two descriptions of a computable group, G, its *index set* and *Scott sentence*. The complexity of these descriptions provides a measure of the computational complexity of the group. In order to introduce these notions we require the following list of basic definitions and facts.

- We call a function, f, a *partial computable function* if it is computable, or halts, on some subset of ω. We say it is total if it halts on every element of ω. If $f(n)$ halts at stage s we denote this by $f(n)[s]\downarrow$.
- A set A is *computable* if there is some total computable function f such that $f(n) = 1$ when $n \in A$ and $f(m) = 0$ if $m \notin A$.
- Every *c.e. set*, A, is the domain of some partial computable function. In fact, it is equivalent to state that A is the range of some partial computable function.
- One can fix a computable enumeration $(\varphi_e)_{e\in\omega}$ of the partial computable functions. We define W_e to be the domain of φ_e. Thus the corresponding list of all the W_e's is a list of all of the c.e. sets.
- We define a *language* $\mathcal{L}$ to be a collection of constants, functions, and relations. Note that the functions and relations are defined to take a certain number of inputs.
- *Formulas* in $\mathcal{L}$ use the symbols of the language, representing the constants, functions, and relations, along with the logical symbols, such as $\exists$ and $\vee$.
- An $\mathcal{L}$-*structure* is a pair (A, I) where A is a set, referred to as the universe, and I is the interpretation of the constants, functions, and relations in $\mathcal{L}$. Specifically, the universe of a group is its underlying set, while the language has one constant, the identity, and the binary operation function of the group.
- We call an $\mathcal{L}$ -structure, $\mathcal{A}$, a *model* for a formula ϕ if ϕ is true in $\mathcal{A}$. Equivalently we state that G *models* ϕ. This is denoted as $G \models \phi$.
- We identify a given structure $\mathcal{A}$ with its *atomic diagram*, denoted $D(\mathcal{A})$, the set of atomic truths in $\mathcal{A}$. Note that atomic truths are by definition quantifier free.

The structures we focus on are computable groups. We will identify a computable group with a specific copy of the group. We will then further identify the copy with its atomic diagram. The atomic diagram of a group is simply the set of equalities and inequalities. We say that a given copy of the group is computable if its atomic diagram is computable.

3.1.1 Infinitary Logic

Typically one uses first-order formulas to describe a given structure.

Example 3.1. *In a group the associative property can be described with the following first-order formula:* $(\forall x)(\forall y)(\forall z)[(xy)z = x(yz)]$.

This particular sentence is written in (finitary) first-order logic. However, for our purposes finitary first order logic does not provide enough power of expression. For instance, as noted earlier, we will investigate torsion-free abelian groups; however, they are not axiomatizable in finitary logic. Intuitively, in order to state that an element has infinite order we would need to state that $x \neq e$, $x^2 \neq e$, and so on such that $x^n \neq e$ for all $n \in \omega$, or some equivalent sentence. The given example seems to require infinite conjunctions, which finitary logic does not allow. In fact, one can show that it is impossible to give an axiomatization of torsion-free abelian groups in finitary logic. However, infinitary logic, denoted $\mathcal{L}_{\omega_1\omega}$ gives us this ability. In addition, as noted before $L_{\omega_1\omega}$ is necessary to distinguish between the sentences satisfied by free groups of differing finite rank. In $\mathcal{L}_{\omega_1\omega}$ one is allowed to take infinite conjunctions and disjunctions, denoted $\bigwedge\!\!\!\bigwedge$ and $\bigvee\!\!\!\bigvee$, respectively. Computable infinitary logic requires that while the conjunctions and disjunctions can be infinite, they must index over a computably enumerable set.

We can now easily state the torsion-free axiom in infinitary logic as follows:
$\forall(x \neq e) \bigwedge\!\!\!\bigwedge_{n\in\omega, n\neq 0} x^n \neq e$.

For more information on $L_{\omega_1\omega}$ see [1].

3.1.2 Formula Complexity

We measure the complexity of a formula using the syntactic hierarchy of computable infinitary formulas. The infinitary formulas, $\phi(\bar{x})$ are described in the following way:

Definition 3.2. *Let* $\phi(\bar{x})$ *be a formula in infinitary logic.*

- $\phi(\bar{x})$ *is* Σ_0 *and* Π_0 *if it is finitary and quantifier free.*
- $\phi(\bar{x})$ *is* Σ_n *if it is an infinite disjunction of formulas of the form* $(\exists\bar{y})\psi(\bar{x},\bar{y})$ *where every* $\psi(\bar{x},\bar{y})$ *is* Π_{n-1}.
- *Similarly,* $\phi(\bar{x})$ *is* Π_n *if it is an infinite conjunction of formulas of the form* $(\forall\bar{y})\psi(\bar{x},\bar{y})$ *where every* $\psi(\bar{x},\bar{y})$ *is* Σ_{n-1}.

Again, since we will be using *computable* infinitary logic we restrict the infinite disjunctions and conjunctions to be computably enumerable. A formula that is both Σ_n and Π_n is called Δ_n. We also define a d-Σ_n set to be the conjunction of a Σ_n set and a Π_n set. Let Γ be some complexity class, such as Σ_3 or d-Σ_2. We will call a set A is Γ, or $A \in \Gamma$, if there is a Γ formula that defines this set.

Note that this computable infinitary hierarchy, based purely on syntax, corresponds to the hierarchy of the relative computability of a given set. This correspondence can be seen in the fact that the Δ_1 sets are exactly the computable sets and the c.e. sets are exactly those defined by Σ_1 formulas. Recall that we have an enumeration $\{\varphi_e\}_{e\in\omega}$ of all the partial computable functions.

Example 3.3. *The* halting set*, $K = \{e \mid \varphi_e(e)$ exists $\}$, is a canonical example of a set that is c.e and not computable. This set is Σ_1 since it is described by the formula $(\exists s)\varphi_e(e)[s]\downarrow$.*

If we have a set W and x is in W at stage s we will denote this by $x \in W[s]$.

We will use the complexity of a formula that defines the group as an upper bound for the complexity of the index set. We have the following notion providing a lower bound on the complexity of a set.

Definition 3.4. *Let Γ be some complexity class. We say that a set, A is Γ-hard if for all $B \in \Gamma$ there exists a computable function $f : \omega \to \omega$ where $f(n) \in B$ if and only if $n \in A$. We call a set Γ-complete if the set is both Γ and Γ-hard.*

3.1.3 Index Set and Scott Sentence

Definition 3.5. *A computable index of a structure $\mathcal{A}$ is an $e \in \omega$ such that $\varphi_e = \chi_{D(\mathcal{A})}$. The* index set *of $\mathcal{A}$ is the set of all such computable indices, denoted $I(\mathcal{A}) = \{e \mid e$ is a computable index of $\mathcal{A}\}$.*

The index set can be thought of as the collection of descriptions of a group. We investigate the complexity of these index sets for various groups.

Another way to describe a given group is with a *Scott sentence*.

Definition 3.6. *A Scott sentence of a countable $\mathcal{L}$-strucuture, $\mathcal{A}$, is a $\mathcal{L}_{\omega_1\omega}$ formula, $\psi(\bar{x})$, such that the isomorphic copies of $\mathcal{A}$ are exactly the countable models of $\psi(\bar{x})$.*

Scott showed that any countable $\mathcal{L}$-structure has such a sentence [11]. A Scott sentence provides an upper bound on the complexity of the index set of $\mathcal{A}$. It was previously thought in the field that the complexity of the index set and the least complex Scott sentence were the same. However, Knight and McCoy recently showed that this is not the case [7].

3.2 Finitely Presented Groups

We examine a variety of groups including finitely presented groups. This expands on previous work investigating the index sets of free groups.

Definition 3.7. *The* free group of rank n *is the group of words on the letters $(a_1, \ldots a_n, a_1^{-1}, \ldots a_n^{-1})$ with the operation of concatenation.*

Definition 3.8. *Let F_n be the free group of rank n generated by $(a_1, \ldots, a_n)$. Let $W = \{w_1, \ldots, w_m\}$ be a subset of F_n and let N be the smallest normal subgroup containing W. We say that $F_n/N \cong G$ is a group with generators $(a_1, \ldots a_n)$ and relations $w_1 = \ldots w_m = 1$. We denote G as $\langle a_1, \ldots a_n | w_1 = \ldots w_m = 1\rangle$.*

Note that this group can be thought of as words on $(a_1, \ldots a_n)$ and their inverses such that any appearance of $w_i \in W$ reduces to 1.

Definition 3.9. *A group G is finitely presented if there exists a presentation of G with a finite number of generators and a finite number of relations.*

Carson et.al., in [3] found that the index set of the free group of rank n is d-Σ_2 complete. Later, it was shown, by Saraph, in [10] that the index set of finitely generated abelian groups and of the infinite dihedral group are both d-Σ_2 complete. In order to expand on these results we investigate certain finitely presented groups. We will examine the complexity of the index sets of groups with presentation of the form: $G = \langle x_1, \ldots, x_m, x_{m+1}, \ldots, x_n \mid x_1^2 = \ldots = x_m^2 = 1\rangle$. This section will address the necessary background.

Definition 3.10. *Let G be a finitely presented group and U be some tuple of words in G. Let Gp(U) denote the group generated by the words in U. We call a tuple U a* generating set *of G if Gp(U) $= G$.*

Note that any generating set of a free group of rank n has at least n elements.

One of the primary tools used to find an upper bound on the complexity of the index set is Nielsen transformations. Nielsen transformations allow us to alter a given tuple of words without changing the group generated by the tuple. We will specifically use these transformation to reduce the total word length of a tuple without changing the group it generates. Note that throughout this thesis we will denote the length of a word w as $|w|$.

Definition 3.11 (Schupp [9])**.** *Let G be a finitely presented group and let $\{a_1, \ldots, a_n\}$ be a subset of G. An* elementary Nielsen transformation *is one of the following:*

1. *Replace some a_i by a_i^{-1};*
2. *Replace some a_i by $a_i a_j$ where $j \neq i$;*
3. *Delete some a_i where $a_i = 1$.*

A Nielsen transformation *is a product of elementary Nielsen transformations.*

We say that if one tuple can be carried to another via a Nielsen transformation then they are *Nielsen equivalent*.

Proposition 3.12 (Schupp [9])**.** *Let U and V be two subsets of a group G. If U is Nielsen equivalent to V, then Gp(U) = Gp(V).*

For more information on Nielsen transformations consult [9]. Since Nielsen transformations allow us to alter a particular tuple of words in a group without altering the words that are generated by the tuple, we can use Nielsen transformations to obtain equivalent tuples that have shorter total word length. Let ϵ denote the empty word, which is the identity element in G.

Definition 3.13 (Schupp [9])**.** *Let G be a finitely generated group and let $U = \{u_1, \ldots, u_n\}$ be a n-tuple of words in G. Consider elements $v_1, v_2, v_3 \in U^{\pm 1}$. Define the following conditions*

(N0) There is no $u_i = \epsilon$.

(N1) If $v_1 v_2 \neq 1$ then $|v_1 v_2| \geq |v_1|$ and $|v_1 v_2| \geq |v_2|$.

(N2) If $v_1 v_2 \neq 1$ and $v_2 v_3 \neq 1$ then $|v_1 v_2 v_3| > |v_1| - |v_2| + |v_3|$.

If a tuple satisfies (N0), (N1), and (N2) then we call it N-reduced.

In the case of free groups all tuples are Nielsen equivalent to an N-reduced tuple [9]. We can show the following related but weaker condition for when a tuple is N-reduced. The following Lemma is a slight generalization of Proposition 2.2 in [9].

Lemma 3.14. *Let U be a tuple of words in the group $G = \langle x_1, \ldots, x_m, x_{m+1}, \ldots, x_n \mid x_1^2 = \ldots = x_m^2 = 1\rangle$, If U satisfies (N0) and (N1) then it is Nielsen equivalent to some tuple that is N-reduced.*

Proof. Assume that U satisfies (N0) and (N1). Note that in G any word has the same length as its inverse.

Take some triple $x, y, z \in U^{\pm 1}$ where $xy \neq 1$ and $yz \neq 1$. Since U is (N1) we know that no more than half of x and y cancel in xy and similarly no more than half of y and z cancel in yz. Thus let $x = ap^{-1}$, $y = pbq^{-1}$, and $z = qc$ where all products are reduced as much as possible by the relations in G. Clearly $xy = abq^{-1}$ and $yz = pbc$.

First assume that $b \neq 1$. Then we have that $xyz = abc$ which is as reduced as possible. Thus $|xyz| = |a|+|p|-|p|+|b|+|q|-|q|+|c| = |x|-|y|+|z|+2|b|$. Since $|b| \geq 1$ we know that $|xyz| > |x|-|y|+|z|$. Therefore the (N2) condition holds in this case.

Now suppose that $b = 1$. Thus we have $x = ap^{-1}$, $y = pq^{-1}$, and $z = qc$. In this case (N2) is violated since $|xyz| = |x| - |y| + |z|$. Thus we must construct a tuple that is Nielsen equivalent and is N-reduced. We will require a two-step process, repeated multiple times, to obtain a Nielsen equivalent N-reduced tuple. The first step will create a tuple satisfying (N2); however, the new tuple may now violate (N1). Thus the second step will created a Nielsen equivalent tuple satisfying (N1). Note that there is some tuple that is Nielsen equivalent to U that has smallest total word length. In our construction step 1 will not alter the total word length, while step 2 will make it strictly smaller. Thus this process will terminate at some point after step 1 leaving a Nielsen equivalent N-reduced tuple.

Recall that $|p| = \frac{1}{2}|y| = |q|$. In this case we have $q \neq p$ since we know that if $q = p$ then $y = pp^{-1} = \epsilon$ contradicting the (N0) condition. Furthermore $|p| \leq \frac{1}{2}|x|$ and $|p| \leq \frac{1}{2}|z|$. We will define a well-ordering in order to determine what Nielsen transformation to apply to this tuple. First take a well-ordering, denoted $<$, on the letters that generate G and their inverses. This clearly induces a lexicographic well-ordering of the words in G. Define the left half of a word $w \in G$ to be the initial segment of length $[\frac{|w|+1}{2}]$, denoted $L(w)$. Now define a well-ordering on pairs of words (w, w^{-1}) as follows. Say $(w_1, w_1^{-1}) \prec (w_2, w_2^{-1})$ if $\min\{L(w_1), L(w_1^{-1})\} < \min\{L(w_2), L(w_2^{-1})\}$ or if the minimums are equal then $\max\{L(w_1), L(w_1^{-1})\} < \max\{L(w_2), L(w_2^{-1})\}$. We can simply write $w_1 \prec w_2$ if $(w_1, w_1^{-1}) \prec (w_2, w_2^{-1})$. Note that this is a global well-ordering on all the pairs of words in G that can be used to determine what Nielsen tranformation to apply to each tuple.

If $p < q$ then note that $yz = pc \prec qc = z$. If this is the case then replace $z = qc$ with $yz = pc$ via a Nielsen transformation of type 2. Note that since $|q| = |p|$ this transformation does not alter $\sum |u_i|$. Conversely, if $q < p$ then replace x with xy again via a Nielsen transformation of type 2. Similarly this transformation does not alter $\sum |u_i|$. Continue to reduce the rank of the u_i under the relation $u \prec u'$ as far as possible. Note that clearly (N0) must hold. In addition, the transformation from (x, y, z) to (xy, y, z) or (x, yz, z) causes this triple to now be in the first case where $b \neq 1$. Thus this triple satisfies the (N2) condition.

However, the new tuple may violate the (N1) condition. In this case assume that there are words v_1 and v_2 in the tuple such that $|v_1 v_2| < |v_1|$. Replace v_1 with $v_1 v_2$. Note that this causes the total word length of the tuple to reduce. Repeat this process until the tuple satisfies (N1). There is a lower bound on the total word length, so this process will terminate. Once we have obtained a Nielsen equivalent tuple satisfying (N0) and (N1) repeat the process of altering the tuple to reach a Nielsen equivalent tuple satisfying (N2). Since the process of altering the tuple to satisfy (N1) reduces the total word length and the process of altering a tuple to satisfy (N2) does not change the total word length, we know this process terminates in a N-reduced tuple that is Nielsen equivalent to the original tuple.

□

We will also require the following similar lemma.

Lemma 3.15. *Let U be a tuple of words in the group $G = \langle x_1, \ldots, x_m, x_{m+1}, \ldots, x_n \mid x_1^2 = \ldots = x_m^2 = 1 \rangle$. If every Nielsen equivalent tuple of U satisfies (N0) then there is a Nielsen equivalent tuple of U that is N-reduced.*

Proof. Assume that U does not satisfy the (N1) condition. Then there is a pair of words $v_1, v_2 \in U^{\pm 1}$ such that $v_1 v_2 \neq 1$ but $|v_1 v_2| < |v_1|$. Replace v_1 with $v_1 v_2$ by a Nielsen transformation of type 2. Note that this reduces the total word length of the tuple. Continue to do these replacements for every pair of words violating the (N1) condition. Since every such transformation reduces the total word length, and there is a lower bound on the total word length, we know that this process terminates. Furthermore, it must terminate in a tuple that is Nielsen equivalent to the original tuple, but also satisfies (N0) and (N1). Thus by Lemma 3.14 it is Nielsen equivalent to some N-reduced tuple. □

Saraph used the following definitions and theorem to find the complexity of the index set of the infinite dihedral group. We will similarly use them to find the complexity of the index set of certain more general finitely presented groups.

Definition 3.16. *[10] Let G be a finitely presented group. An n-tuple of words on the variables $\bar{x}$, denoted $\bar{w}(\bar{x})$, is* primitive *if for a generating set $\bar{a}$ of G, we have that $\bar{w}(\bar{a})$ is also a generating set for G.*

Example 3.17. *The following are primitive tuples in F_n.*

1. *The identity tuple $(x_1, \ldots, x_n)$ or any permutation of this tuple.*
2. *The tuple $(x_1 x_2 x_5, x_2, x_3, \ldots, x_n)$ if $n \geq 5$.*
3. *The tuple $(x_1 x_3^{-1}, x_2 x_5, x_3, x_4, x_5, \ldots, x_n)$ if $n \geq 5$.*

Example 3.18. *The following are tuples in F_n that are not primitive.*

1. *The tuple $(w_1, \ldots w_n)$ where some $w_i = \epsilon$.*
2. *The tuple $(x_1, x_3 x_4, x_2 x_4, x_5, \ldots x_n)$ if $n \geq 5$.*
3. *The tuple $(x_1 x_3, x_2, x_2, x_4, \ldots, x_n)$ if $n \geq 5$.*

Definition 3.19. *[10] Let G be a finitely presented group, with presentation $G = \langle \bar{a}, R \rangle$. Define $Rel(R) = \{R'(\bar{x}) = R(\bar{w}(\bar{x})) : \bar{w}$ is primitive $\}$.*

Clearly there are many different representations of a given group. We start by stating G in terms of the relations R. The relations $R' \in Rel(R)$ are exactly those relations that act on $\bar{x}$ in the same manner as first applying a primitive tuple $\bar{w}$ and then the original relations R.

Example 3.20. *Let $G = \langle a_1, a_2 | a_1^2 = 1 \rangle$. Define R to be the relation $a_1^2 = 1$. Let R' be the relation $a_2^2 = 1$. Clearly we know that $\bar{w} = (x_2, x_1)$ is primitive. Thus we have that for all (y_1, y_2) that $R'(y_1, y_2)$ states that $y_2^2 = 1$. Similarly $R(\bar{w}(y_1, y_2)) = R(y_2, y_1)$ again stating that $y_2^2 = 1$. Hence, we have $R' \in Rel(R)$.*

Definition 3.21. *[10] Let $G = \langle \bar{a}, R \rangle$. Let $R' \in Rel(R)$. We say $\bar{w}$ is primitive relative to R' if $R'(\bar{w})$ is some primitive tuple $\bar{v}$. Define $Imp(R') = \{\bar{w} : \bar{w}$ is not primitive relative to $R'\}$.*

Saraph used these two properties to create the following conditions for when G has a d-Σ_2 index set.

Theorem 3.22. *(Saraph) [10] Suppose G is a finitely generated group with presentation $\bar{a}, R$. If:*

1. *Any two generating sets of minimal size are Nielsen equivalent;*
2. *For $R' \in Rel(R)$, $Imp(R')$ is computably enumerable, uniformly in R'.*

Then G has a Scott sentence that is computable d-Σ_2, so $I(G)$ is d-Σ_2.

The following proposition follows directly from Theorem 3.22 and is implicit in the work by Saraph.

Observation 3.23. *If for a given finitely presented group G, the set of primitive tuples is computable and any two generating sets of minimal size are Nielsen equivalent, then the index set of G is d-Σ_2.*

Proof. Assume that the set of primitive tuples in G is computable. Then we know that G satisfies property 2 of Theorem 3.22. We assume that the first property is satisfied, and thus by Theorem 3.22 we know that $I(G)$ is d-Σ_2.

□

3.3 Torsion-Free abelian Groups

We now explore the torsion-free abelian groups of rank 1.

Definition 3.24. *A* torsion-free abelian group *is an abelian group such that no element except for the identity has finite order. A torsion-free abelian group of rank 1 is defined to be a torsion-free abelian group with the additional property that for any two elements, a and b, there exists $n, m \in \mathbb{Z}$ such that $na + mb = 0$.*

A torsion-free abelian group embeds in a vector space over $\mathbb{Q}$ whose dimension is the rank of the group. Thus a torsion-free abelian group of rank 1 can be thought of as a subgroup of $\mathbb{Q}$. Torsion-free abelian groups of finite rank are similar to finitely generated groups since there are finitely many basis elements who in a certain sense generate the entire group. We focus on the rank 1 torsion-free abelian groups.

It is not hard to see that any torsion-free abelian group has a Σ_3 Scott sentence, and thus we know that none of these groups will have index sets of more complex then Σ_3. Such a Scott sentence can be found in [10]. As done by Saraph, we will split these groups into a variety of cases depending on the divisibility of the primes. Fix some non-zero element and denote it 1.

Definition 3.25 (Saraph [10]). *Let $G \subseteq \mathbb{Q}$ be a subgroup, and let P denote the set of prime numbers. We say that $n|a$ for $a \in G$ if there exists a $b \in G$ such that $nb = a$. Then we partition P as $P = P^0 \cup P^{fin} \cup P^\infty$ where:*

1. $P^0 = \{p \in P : G \models p \nmid 1\}$,
2. $P^{fin} = \{p \in P : G \models p^k|1$ *and* $p^{k+1} \nmid 1$ *for some* $k > 0\}$, *and*
3. $P^\infty = \{p \in P : G \models p^k|1$ *for all* $k > 0\}$.

Regardless of the group it is straightforward to check that P^0 is Π_1, P^{fin} is Σ_2, and P^∞ is Π_2. If we choose a different non-zero element for 1 then only a finite number of primes can switch from one category to another. Since we will only be considering these categories relative to their complexity, changes of a finite amount of information are acceptable. Saraph divided the torsion-free abelian groups of rank 1 into seven cases based on whether the various subsets of primes, P^0, P^{fin}, and P^∞, are finite or infinite, and explored the index set complexity of these cases. We will address a few of the open cases.

In order to show that certain kinds of torsion-free abelian groups have a Σ_3 index set, we will use Theorem 3.27 (Levi's Theorem). This theorem gives a characterization of how different the prime divisibility of 1 can be in two isomorphic groups. First we define the Baer sequence of a torsion-free abelian group of rank one. This sequence encodes divisibility information of every prime in our given group G.

Definition 3.26 (Baer [2]). *Let G be a torsion-free abelian group of rank 1. Fix a non-zero element $h \in G$, and let $\{p_i\}_{i\in\omega}$ be the list of primes in increasing order. The* p-height *of h is defined to be:*

$$ht_p(h) = \begin{cases} k & \text{if } k \text{ is the largest natural number such that } p^k \text{ divides } h \\ \infty & \text{if } p^k \text{ divides } h \text{ for all } k \end{cases}$$

The Baer sequence *with respect to h, denoted $B_{G,h}$, is a function of the form $B_{G,h} : \omega \to \omega \cup \{\infty\}$ such that $B_{G,h}(n) = ht_{p_n}(h)$. We define an equivalence relation, $\sim$, on such functions $f : \omega \to \omega \cup \{\infty\}$ as follows. $f \sim g$ if $f(n) \neq g(n)$ for only finitely many n and neither $f(n)$ nor $g(n)$ is equal to ∞ when $f(n) \neq g(n)$.*

Note that if h and h' are both non-zero elements of G then $B_{G,h} \sim B_{G,h'}$. We will let B_G be some representative of this equivalence class. Levi's Theorem characterizes when torsion-free abelian groups of rank 1 are isomorphic.

Theorem 3.27 (Levi [8])**.** *Two rank one torsion-free abelian groups, G and G', are isomorphic if and only if $B_G \sim B_{G'}$.*

Thus we know that two torsion-free abelian groups of rank one, G and G', are isomorphic exactly when the primes in P^∞ of G and G' are equal and there are only a finite number of primes in $P^0 \cup P^{fin}$ that have a different power dividing 1.

4 Finitely Generated Groups

In this section we will show that groups with presentation of the form
$G = \langle x_1, \ldots, x_m, x_{m+1}, \ldots, x_n | x_1^2 = \ldots = x_m^2 = 1\rangle$ have d-Σ_2 complete index sets. We first show in the following lemma that this class of groups is d-Σ_2 hard.

4.1 Hardness

Lemma 4.1. *If G is a group with presentation*
$G = \langle a_1, a_2, \ldots a_m, a_{m+1}, \ldots, a_n | a_1^2 = a_2^2 = \ldots = a_m^2 = 1\rangle$*, then $I(G)$ is d-Σ_2 hard.*

Proof. To show that $I(G)$ is Σ_2 hard we will show that every d-Σ_2 set can be encoded in $I(G)$. Let $S = S_1 - S_2$ where S_1 and S_2 are both Σ_2 sets. We define H to be an infinitely generated group, with generators $\{a_2, \ldots, a_k, \ldots a_l\}$ and $\{b_i\}_{i\in\omega}$, such that $b_i = b_{i+1}a_2b_{i+1}$ where $b_1 = a_1$. Define h_n to be a partial computable function, which only computes a finite fragment of $D(G)$. We will construct a computable list of indices $(j_n)_{n\in\omega}$ for partial computable functions such that

$$\varphi_{j_n} = \begin{cases} \chi_{D(H)} & n \notin S_1 \\ \chi_{D(G)} & n \in S_1 - S_2 \\ h_n & n \in S_1 \cap S_2 \end{cases}$$

Recall that $S_1[s]$ is the set of elements in S_1 at stage s, and $S_2[s]$ is similarly defined. The following statement describes the stage approximations of a Σ_2 set. Let $S_1[s]$ and $S_2[s]$ be computable approximations of S_1 and S_2 respectively such that

$$n \in S_i \text{ if and only if } n \in S_i[s] \text{ for all but finitely many } s.$$

We start by assuming that $n \in S_1[0] - S_2[0]$, so our target structure is G at stage 0. Thus we start with generators $a_1, \ldots, a_k$. If there is no change to the target structure at stage $s+1$ then we continue to build the target structure. Now suppose the target structure changes. If $n \notin S_1[s]$ but $n \in S_1[s+1]$ we will simply continue to build according to $D(G)$. If $n \in S_1[s]$ but $n \notin S_1[s+1]$, then we take the smallest i such that b_i is a generator, and let a fresh element, b_{i+1}, be a new generator defined as follows: $b_i = b_{i+1}a_2b_{i+1}$. Thus we now add the statement, $b_i = b_{i+1}a_2b_{i+1}$, into the atomic diagram of the group. If $n \in S_1[s]$ for an infinite number of stages, then we will produce a group, H, that is infinitely generated and thus not isomorphic to G. If $n \in S_1 - S_2$ then we will eventually stop returning to H. This group will be generated by $\{a_2, \ldots, a_l, b_j\}$, where b_j is the last fresh b_i to be taken as a generator. By construction we know that b_j has order two, and thus this group is isomorphic to G.

Now assume that our structure changes because of S_2. In this case we will simply adjust by stopping and continuing the construction of G as necessary. Let $n \notin S_2[s]$ but $n \in S_2[s+1]$. Then we will stop adding to $D(G)$. Conversely if $n \in S_2[s]$ but $n \notin S_2[s+1]$ then we will continue to build $D(G)$ from where we left off. If $n \in S_1[s] - S_2[s]$ infinitely often, then we will eventually construct $D(G)$. However, if $n \in S_1[s] \cap S_2[s]$ then we will only construct a finite fragment of G. Thus we obtain the appropriate indices j_n, since $\varphi_{j_n} = \chi_{D(G)}$ exactly when $n \in S_1 - S_2$. Hence, G is d-Σ_2 hard. ☐

Therefore, we have that the complexity of $I(G)$ is at least d-Σ_2.

4.2 Completeness

We will now show that the index set of G is in fact d-Σ_2, using Theorem 3.22 and Observation 3.23. Thus we must prove that the set of primitive tuples of G is computable. The following are two propositions that will be used to develop an algorithm for determining when a tuple is primitive.

Proposition 4.2 (Schupp [9])**.** *Let U be an N- reduced tuple of words in G. Then one may consider each $u \in U^{\pm 1}$ to be of the form $a(u)m(u)a(u^{-1})^{-1}$ reduced where $m(u) \neq 1$ such that if $w = u_1 \ldots u_t$ for $t \geq 0$, $u_i \in U^{\pm 1}$, and all $u_i u_{i+1} \neq 1$ then the $m(u_1), \ldots, m(u_t)$ remain uncancelled in the reduced form of w.*

The following proposition is slightly expanded from Proposition 2.13 in [9].

Proposition 4.3. *Let U be an N-reduced subset of $G = \langle x_1, \ldots, x_m, x_{m+1}, \ldots, x_n | x_1^2 = \ldots = x_m^2 = 1\rangle$ and $w = u_1 \ldots u_n$ where each $u_i \in U^{\pm 1}$ and no $u_i u_{i+1} = 1$. Then $|w| \geq |u_1|, \ldots, |u_n|$.*

Proof. By Proposition 4.2 we have that no u_i is completely cancelled in w. Clearly $|u_i| \leq |u_i u_{i+1}|$ or else the tuple would violate the (N1) condition. Assume for induction that $|u_i| \leq |u_i \ldots u_{i+k}|$. Note that the part of u_{i+k} that cancels in $u_i \ldots u_{i+k+1}$ is the same part of u_{i+k} that cancels in $u_{i+k}u_{i+k+1}$. Furthermore, this part is no larger than the uncanceled portion of u_{i+k+1}. Thus we have that $|u_i| \leq |u_i \ldots u_{i+k}| \leq |u_i \ldots u_{i+k+1}|$. Hence, we have that $|u_i| \leq |u_i \ldots u_n|$. Similarly by concatenating on the left side we have that $|u_i| \leq |u_i \ldots u_n| \leq |u_{i-1} \ldots u_n| \leq |u_1 \ldots u_n| = |w|$. ☐

We have one short lemma to aid in the proof of an algorithm for determining when a tuple is primitive in G.

Lemma 4.4. *Let $G = \langle x_1, \ldots, x_m, x_{m+1}, \ldots, x_n | x_1^2 = \ldots = x_m^2 = 1\rangle$ and let $W = (w_1, \ldots, w_n)$ be a tuple of words on variables such that some word in the tuple has length at least 2. If for every $z = v_1 \ldots v_k$ where $v_i \in W^{\pm 1}$ we have that $|z| \geq |v_1|, \ldots, |v_k|$ then we have that W is not primitive.*

Proof. Note that a generating set of G is $\bar{x} = (x_1, \ldots x_n)$. However, we have that $(w_1(\bar{x}), \ldots, w_n(\bar{x}))$ is not a generating set for G. This is because the new tuple must not have some x_i. Furthermore, this x_i cannot be generated by elements in the tuple. We know that x_i has length 1 and thus cannot be generated by a concatenation of words $v_1 \ldots v_k$ where one of the v_i's has length greater than 1 by Proposition 4.3. If every v_k had length 1 they could not equal x_i unless some $v_j = x_i$ because of the given relations of the group. Thus some word of length one is not generated by $(w_1(x_1), \ldots, w_n(x_n))$ so $(w_1(x_1), \ldots, w_n(x_n))$ does not generate G. Thus W is not primitive. □

We use the above propositions and lemmas in the following lemma to show that there is an algorithm for determining if a given tuple is primitive or not.

Lemma 4.5. *There is an algorithm for determining which tuples* $(w_1, \ldots w_n)$ *are primitive in* $G = \langle x_1, \ldots, x_m, x_{m+1}, \ldots, x_n | x_1^2 = \ldots = x_m^2 = 1\rangle$.

Proof. The following are a few simple base cases.

1. A tuple of the form $(w_1, \ldots, w_n)$ where some w_i is the empty word is not primitive.
2. A tuple of the form $(w_1, \ldots, w_n)$ where each $x_i = w_j$ for exactly one j is primitive.
3. A tuple of the form $(w_1, \ldots, w_n)$ is not primitive if it satisfies the requirements of Lemma 4.4.

Assume by induction that if some tuple has total word length less than or equal to k then we can determine if the tuple is primitive or not.

Now take some tuple $W = (w_1, \ldots w_m)$ whose total word length is $k + 1$, and is not one of the base cases.

Case 1: First consider the case where there is some $w_i = w_j^{-1}$ where $i \neq j$. Then replace w_i with $w_i w_j = \epsilon$ via a Nielsen transformation of type 2. Then we have a Nielsen equivalent tuple that is in the first base case, so the original tuple is not primitive. Similarly if $w_i = w_j$ for $i \neq j$ we can again obtain a Nielsen equivalent tuple of the form of the first base case. Thus the original tuple is not primitive. Now assume there are no such pairs of words, but there is some pair of words v_1, v_2 of the form $w_i^{\pm 1}$ where $|v_1 v_2| < |v_1|$ or $|v_1 v_2| < |v_2|$ and $v_1 \neq v_2^{-1}$. Then replace the larger of v_1 and v_2 with $v_1 v_2$ to achieve a Nielsen equivalent tuple with a shorter total word length. Thus by our induction hypothesis we can determine if the tuple is primitive or not.

Case 2: There is no pair of words v_1, v_2 of the form $w_i^{\pm 1}$ where $|v_1 v_2| < |v_1|$ or $|v_2|$. This is equivalent to the statement that W satisfies the (N0) and (N1) conditions. Furthermore, by Lemma 3.14 we know that W is Nielsen equivalent to some N-reduced tuple W'. If the total word length of W' is less than W then we can apply our induction hypothesis to determine whether W is primitive. If not then by Proposition 4.3 we have that every word $z = v_1 \ldots v_k$ where each $v_i \in W'^{\pm 1}$ has length greater than or equal to $|v_i|$ for all i. Thus by Lemma 4.4 we know that W' and thus W are not primitive.

□

Note that the tools used to construct this result relied primarily on the fact that every word in $G = \langle x_1, \ldots, x_m, x_{m+1}, \ldots, x_n | x_1^2 = \ldots = x_m^2 = 1\rangle$ has the same length as its inverse. Thus for any other group where this fact holds there is a possibility these tools can be used similarly. However, in groups where this is not the case our method does not apply.

Theorem 4.6. *Let $G = \langle x_1, \ldots, x_m, x_{m+1}, \ldots, x_n | x_1^2 = \ldots = x_m^2 = 1\rangle$. Then $I(G)$ is d-Σ_2 complete.*

Proof. By Lemma 4.1 we know that $I(G)$ is d-Σ_2-hard. Furthermore, we know by Lemma 4.5 that there is an algorithm for determining what tuples are primitive in G. Let U be a generating tuple of G. Clearly every Nielsen equivalent tuple must also generate G. Since these tuples generate G they must satisfy the (N0) property. By Lemma 3.15 we know that this tuple must be equivalent to a N-reduced tuple. However, by Proposition 4.2 and Lemma 4.4 we know that any N-reduced tuple that is not Nielsen equivalent to the identity tuple cannot generate all of G. Thus the original tuple must be equivalent to the identity tuple. Hence, any two generating tuples are Nielsen equivalent. Therefore by Proposition 3.23 $I(G)$ is d-Σ_2 complete.

□

5 Torsion-free abelian Groups

5.1 Previous Results

In [10] Saraph obtained the following results on the complexity of index sets of various torsion-free abelian groups of rank 1. The following table compiles the known upper and lower bounds on the index sets of torsion-free abelian groups with the corresponding description type.

Theorem 5.1 (Saraph [10])**.**

Case	*Description*	*Lower Bound*	*Upper Bound*
1	*P^0 is infinite, P^{fin} is finite, and P^∞ is finite*	d-Σ_2	d-Σ_2
2	*P^0 is finite, P^{fin} is infinite, and P^∞ is finite*	Σ_3	Σ_3
3	*P^0 is finite, P^{fin} is finite, and P^∞ is infinite*	d-Σ_2	d-Σ_2
4	*P^0 is finite, P^{fin} is infinite, and P^∞ is infinite*	d-Σ_2	Σ_3
5	*P^0 is infinite, P^{fin} is finite, and P^∞ is infinite*	d-Σ_2	Σ_3
6	*P^0 is infinite, P^{fin} is infinite, and P^∞ is finite*	Σ_3	Σ_3
7	*P^0 is infinite, P^{fin} is infinite, and P^∞ is infinite*	d-Σ_2	Σ_3

Note that Saraph found completeness results for Cases 1, 2, 3, and 6. However, Cases 4, 5, and 7 were left open. Saraph proved the following lemma giving an upper bound on the complexity of all torsion-free abelian groups of finite rank.

Lemma 5.2 (Saraph [10])**.** *Let G be a computable torsion-free abelian group of finite rank. Then there is a computable Σ_3 Scott sentence describing G, and hence $I(G)$ is Σ_3.*

Saraph also proved that the following condition guarantees that an index set is Σ_3-hard.

Lemma 5.3 (Saraph [10]). *Let G be a torsion-free abelian group of rank 1, with P^{fin} having an infinite computable subset, then $I(G)$ is Σ_3-hard.*

However, we know that generally P^{fin} is Σ_2 and not every Σ_2 set contains an infinite computable subset.

Example 5.4. *A canonical example of such a set is an* immune *set. Immune sets are defined to be Π_1 sets with no infinite c.e subsets. Clearly since an immune set is Π_1 it is also Σ_2. Additionally, since every infinite computable set is also c.e we know that an immune set cannot contain any infinite computable subsets.*

Immune sets are often characterized in terms of their complements, the *maximal* sets. [20]

We expand on these results in the following section by investigating the cases where the upper and lower bounds differ.

5.2 Results

5.2.1 Groups with index sets that are d-Σ_2 complete

We first address case 5 where we have that P^0 is infinite, P^{fin} is finite, and P^∞ is infinite. We know that in general P^0 is Π_1, so in this case P^∞ must be Σ_1.

Lemma 5.5. *If G is a computable torsion-free abelian group of rank 1 such that P^0 is infinite, P^{fin} is empty, and P^∞ is infinite, then $I(G)$ is d-Σ_2.*

Proof. Since P^0 is Π_1 suppose $P^0 = \{p \mid (\forall \bar{t})R(\bar{t}, p)\}$. Similarly we know that since P^{fin} is empty we have P^∞ is the complement of P^0. Thus P^∞ is defined by the negation of $(\forall \bar{t})R(\bar{t}, p)$ we denote this Σ_1 sentence to be $(\exists \bar{z})Q(\bar{z}, p)$. The following is a d-Σ_2 Scott sentence describing the group:

$$(\forall y, p, \bar{z}, r) \bigwedge\!\!\!\bigwedge_{k\in\omega} (\exists s, \bar{t})[[\neg Q(\bar{z}, p) \vee (sp^k = y)] \wedge [(rp^k \neq y) \vee Q(\bar{t}, p)]] \tag{1}$$

$$\bigwedge \tag{2}$$

$$(\exists x)(\forall p, s, \bar{t})[\neg(sp \neq x) \vee R(\bar{t}, p)] \tag{3}$$

Consider the Π_2 conjunct (1). The first conjunct says that if a prime satisfies $(\exists \bar{z})Q(\bar{z}, p)$ then every element is infinitely divisible by it. The second conjunct requires that if a prime infinitely divides every element then it cannot satisfy $(\forall \bar{t})R(\bar{t}, p)$. Specifically if a prime is in P^∞ then there is always some r such that $rp^k = y$. Hence it is required that $\neg R(\bar{t}, p)$ is true for p.

The Σ_2 conjunct, (3), states that if the stated element x is not divisible by a prime then that prime must satisfy $(\forall \bar{t})R(\bar{t}, p)$. Hence, we know that an element is in P^0 exactly when it satisfies $(\forall \bar{t})R(\bar{t}, p)$ and similarly an element is in P^∞ exactly when it satisfies $(\exists \bar{z})Q(\bar{z}, p)$. Thus this sentences describes the given group and is clearly d-Σ_2. Hence $I(G)$ is d-Σ_2.

□

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The following corollary is a slight expansion of the lemma, allowing P^{fin} to be finite.

Corollary 5.6. *If G is a computable torsion-free abelian group of rank 1 such that P^0 is infinite, P^{fin} is finite, and P^∞ is infinite, then $I(G)$ is d-Σ_2.*

Proof. We can use the same equation as in Lemma 5.5 with slight alterations. We know P^{fin} is finite so let $P^{fin} = \{p_1, \dots p_n\}$. We still let $(\exists \bar{z})Q(\bar{z}, p)$ define P^∞ but it is now no longer simply the negation of $(\forall \bar{t})R(\bar{t}, p)$, but rather is $(\exists \bar{t})(\neg R(\bar{t}, p)) \wedge \bigwedge_{i=1}^{n} p \neq p_i$. In the sentence we will need to make the following change. In the Π_2 statement, (1), there is a conjunct stating $[(rp^k \neq y) \vee Q(\bar{t}, p)]$. This will be altered to state $[(rp^k \neq y) \vee [Q(\bar{t}, p)] \bigvee_{i=1}^{n}(p = p_i)]$. This change is necessary so that the elements in P^{fin} satisfy (1).

□

We thus obtain a characterization of Case 5.

Theorem 5.7. *If G is a computable torsion-free abelian group of rank 1 such that P^0 is infinite, P^{fin} is finite, and P^∞ is infinite, then $I(G)$ is d-Σ_2-complete.*

Proof. By the Lemma 5.6 we know that the index set is *d*-Σ_2. Furthermore, Saraph showed in Theorem 5.1 that the index set is *d*-Σ_2-hard [10]. Thus the index set is *d*-Σ_2-complete. □

5.2.2 Groups with Σ_3 complete Index Sets

Now we investigate a subcase of Cases 4 and 7. In this case we have that P^{fin} and P^∞ are infinite. Recall that P^{fin} is always Σ_2 and P^∞ is always Π_2.

In order to show that a set is Σ_0^3 hard we must show that a Σ_3 complete set can be encoded into it. Cof is a Σ_3 complete set that is often used for this purpose. We say that a set is *cofinite* if its complement is finite.

Definition 5.8. *Let W_e be the set of inputs for which φ_e halts. We let Cof denote the set of indices for which W_e is cofinite, or Cof$= \{e \mid W_e$ is cofinite $\}$.*

We will require that P^∞ be cohesive.

Definition 5.9. *An infinite set, A, is* cohesive *if for all c.e. sets W_n we have either $A \cap W_n$ is finite or $A \cap \overline{W}_n$ is finite.*

Cohesive sets often appear in the literature of classical computability theory as the complements of maximal sets as in [4]. In addition, they are used in the study of Ramsey theory as in [5].

Lemma 5.10. *If G is a computable torsion-free abelian group of rank 1 such that P_0 is computable and P^∞ is cohesive, then $I(G)$ is Σ_3-hard.*

Proof. We wish to construct a set of groups $\{\mathcal{G}_n\}_{n\in\omega}$ such that $\mathcal{G}_n \cong G$ exactly when $n \in$ Cof.
Construction of $\mathcal{G}_n$: First fix some $1 \in \mathcal{G}_n$. Denote P^0, P^{fin} and P^∞ in G as P^0_G, P^{fin}_G and P^∞_G respectively. Similarly denote P^0, P^{fin} and P^∞ in $\mathcal{G}_n$ as $P^0_{\mathcal{G}_n}, P^{fin}_{\mathcal{G}_n}$ and $P^\infty_{\mathcal{G}_n}$ respectively. Let

$\{p_i\}_{i\in\omega} = P_G^{fin} \cup P_G^{\infty}$. This is computable since its complement, namely P_G^0, is computable. Let $\{p_i\}_{i\in\omega} = P_{\mathcal{G}_n}^{fin} \cup P_{\mathcal{G}_n}^{\infty}$. Note that since $\overline{W}_n$ is Π_1 we can start by assuming that every $k \in \overline{W}_n$ at stage 0.

Let t be the maximum value such that $p_k^t|1$ in G at stage s. If $k \in \overline{W}_n$ at stage s then let $p_k^r|1$ in $\mathcal{G}_n$ for all $r < t$ and $p_k^t \nmid 1$ in $\mathcal{G}_n$. If at some stage we see that k enters W_n then assign p_k to *follow* the least p_m such that $m \in \overline{W}_n$ and p_m does not already have a follower. For p_k to be a follower of p_m means that if $m \in \overline{W}_n$ we let $p_k^r|1$ in $\mathcal{G}_n$ for all $r < t$ and $p_k^t \nmid 1$ in $\mathcal{G}_n$. However if m enters W_n at some stage then let $p_k^r|1$ in $\mathcal{G}_n$ exactly when $p_k^r|1$ in G. Thus the number of powers of p_k that divide 1 is dependent on whether $m \in \overline{W}_n$ or not. Furthermore, if p_k had a follower then when k entered W_n the follower would begin to have the same number of powers dividing 1 in $\mathcal{G}_n$ as in G at every subsequent stage.

Verification: We now must verify that $\mathcal{G}_n \cong G$ exactly when $n \in$ Cof. Recall that by Levi's Theorem 3.27 we have that G and $\mathcal{G}_n$, are isomorphic exactly when $P_G^{\infty} = P_{\mathcal{G}_n}^{\infty}$ and there are only a finite number of primes in $P^0 \cup P^{fin}$ that have a different number of powers dividing 1. In every $\mathcal{G}_n$ we have that $P_G^0 = P_{\mathcal{G}_n}^0$, $P_G^{fin} = P_{\mathcal{G}_n}^{fin}$ and $P_G^{\infty} = P_{\mathcal{G}_n}^{\infty}$. Note that if $p_k \in P^{\infty}$ then p_k will still have an infinite number powers dividing 1 in $\mathcal{G}_n$. This is because the difference in the number of powers dividing 1 in G than $\mathcal{G}_n$ is at most one at any stage. We now consider the primes in P^{fin}.

If $n \in$ Cof then there are only a finite number of $p_k \in P^{fin}$ such that $k \in \overline{W}_n$. More generally, there are only a finite number of $m \in \overline{W}_n$. Recall that $p_k \in P^{fin}$ will have a different number of powers dividing 1 in $\mathcal{G}_n$ than G only when either $k \in \overline{W}_n$ or $k \in W_n$ and p_k is following some p_m such that $m \in \overline{W}_n$. Thus since $\overline{W}_n$ is finite only a finite number of $p_k \in P^{fin}$ will have one less power dividing 1 in $\mathcal{G}_n$ than G. The other primes in P^{fin} will have the exact same number of powers dividing 1 in $\mathcal{G}_n$ as in G. Hence, by Levi's Theorem 3.27 we have that if $n \in$ Cof then $\mathcal{G}_n \cong G$.

Now assume that $n \notin$ Cof, so $\overline{W}_n$ is infinite. Furthermore, since we assume that P^{∞} is cohesive, we know that for all W_n either $P^{\infty} \cap \overline{W}_n$ or $P^{\infty} \cap W_n$ is finite. We wish to show that $\mathcal{G}_n \ncong G$. First assume that $P^{\infty} \cap \overline{W}_n$ is finite. Then there must be an infinite number of primes $p_k \in P^{fin}$ such that $k \in \overline{W}_n$. Thus an infinite number of primes in P^{fin} will have one less power dividing 1 in $\mathcal{G}_n$ than G.

Now assume that $P^{\infty} \cap W_n$ is finite. Thus only finitely many p_m who were assigned a follower have m enter W_n at some stage. If $P^{fin} \cap \overline{W}_n$ is infinite then as in the above case we know that $\mathcal{G}_n \ncong G$. Instead suppose that $P^{fin} \cap \overline{W}_n$ is finite, so there must be an infinite number of primes in $P^{fin} \cap W_n$ since P^{fin} is infinite. Each such $p_k \in P^{fin} \cap W_n$ will be assigned to follow a different element. However, by assumption only finitely many p_m who were assigned a follower have m enter W_n at some stage, so there are infinitely many p_m's with followers that are in $\overline{W}_n$. By construction, every follower of such a p_m has one less power dividing 1 in $\mathcal{G}_n$ than G. Therefore, there are an infinite number of $p_k \in P^{fin}$ such that there are a different number of powers dividing 1 in $\mathcal{G}_n$ than G. Hence, by Levi's Theorem 3.27 if $n \notin$ Cof then $\mathcal{G}_n \ncong G$. So in all cases we have that $\mathcal{G}_n \cong G$ exactly when $n \in$Cof. Thus we have that $I(G)$ is Σ_3-hard. □

Theorem 5.11. *If G is a computable torsion-free abelian group of rank 1 such that P^0 is computable and P^{∞} is cohesive, then $I(G)$ is Σ_3-complete.*

Proof. We know by Lemma 5.10 that the index set is Σ_3-hard. Furthermore, Saraph showed in Lemma 5.2 that all index sets of torsion-free abelian groups of rank 1 are Σ_3 [10]. Thus the index set is Σ_3 complete. □

Below is an updated version of the table of index set complexities for computable torsion-free abelian groups of rank 1.

Theorem 5.12.

Case	*Description*	*Lower Bound*	*Upper Bound*
1	*P^0 is infinite, P^{fin} is finite, and P^∞ is finite*	$d\text{-}\Sigma_2$	$d\text{-}\Sigma_2$
2	*P^0 is finite, P^{fin} is infinite, and P^∞ is finite*	Σ_3	Σ_3
3	*P^0 is finite, P^{fin} is finite, and P^∞ is infinite*	$d\text{-}\Sigma_2$	$d\text{-}\Sigma_2$
4	*P^0 is finite, P^{fin} is infinite, and P^∞ is infinite*	$d\text{-}\Sigma_2^*$	Σ_3
5	*P^0 is infinite, P^{fin} is finite, and P^∞ is infinite*	$d\text{-}\Sigma_2$	$d\text{-}\Sigma_3$
6	*P^0 is infinite, P^{fin} is infinite, and P^∞ is finite*	Σ_3	Σ_3
7	*P^0 is infinite, P^{fin} is infinite, and P^∞ is infinite*	$d\text{-}\Sigma_2^*$	Σ_3

Recall that Theorem 5.11 suggests that there are some additional subcases of Cases 4 and 7 where the lower bound is raised to Σ_3. For more information see Question 6.3 below.

6 Future Work

6.1 Finitely Presented Groups

Question 6.1. *A question of interest is whether every finitely presented group has an index set that is d-Σ_2 complete. Currently, there are no such examples, but the index set complexity of many finitely presented groups remains unknown. Unfortunately, the methods used in this thesis relied on the fact that every word had the same length as its inverse. In the many groups where this no longer holds it may be possible to find a finitely presented group with a more complex index set.*

Question 6.2. *A more specific question of interest is whether every tuple of a finitely present group that satisfies (N0) and (N1) is Nielsen equivalent to an N-reduced tuple. This is an expansion of Lemma 3.14. Again, the tools used in the proof of Lemma 3.14 relied on the fact that a word and its inverse had the same length, and thus may not be more broadly applicable.*

6.2 Torsion-free abelian Groups

Question 6.3. *In Theorem 5.11 we stated that every torsion-free abelian group with P^0 computable and P^∞ cohesive has an index set that is Σ_3 complete. Note that a previous result by Saraph, namely Lemma 5.3, gives that if P^{fin} has an infinite computable subset then the index set is Σ_3. Since every Σ_1 set has an infinite computable subset we know that if P^∞ is Π_1 then the index set is Σ_3.*

We have reason to believe that there are in fact cohesive sets that are Π_2 and not Π_1 making our result non-trivial. Often cohesive sets are thought of in the context of maximal *sets, where the complexity is Π_1. In addition, there are no non computable Σ_1 cohesive sets because any Σ_1 set contains an infinite computable subset. However, because of work by Jockusch in [5] and [6] we believe such a Π_2 cohesive set exists.*

Question 6.4. *Another open question is to determine the index set complexity of the last two cases of the torsion-free abelian groups of rank 1. We know the upper bound is Σ_3, and given that subcases of groups have index sets that are Σ_3-complete we feel it is likely for the index sets to be Σ_3 complete in the larger class of groups. However, as the subsets of primes P^0, P^{fin}, and P^∞ become more complex, the methods used in this thesis fail.*

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