Wellesley College
Wellesley College Digital Scholarship and Archive

Honors Thesis Collection

2013

Towards Enumerations of C-alt and D Matrices

Ran Ji
rji@wellesley.edu

Follow this and additional works at: https://repository.wellesley.edu/thesiscollection

Recommended Citation

Ji, Ran, "Towards Enumerations of C-alt and D Matrices" (2013). *Honors Thesis Collection*. 166.
https://repository.wellesley.edu/thesiscollection/166

This Dissertation/Thesis is brought to you for free and open access by Wellesley College Digital Scholarship and Archive. It has been accepted for inclusion in Honors Thesis Collection by an authorized administrator of Wellesley College Digital Scholarship and Archive. For more information, please contact ir@wellesley.edu.

Towards Enumerations of $\mathcal{C}_n^*$ and $\mathcal{D}_n$ Matrices

Author: Ran Ji

Advisor: Dr. Andrew C. Schultz

Submitted in Partial Fullfillment

of the

Prerequisite for Honors

in the Department of Mathematics

May 19, 2013

© 2012 Ran Ji

Contents

1. Introduction 2

1.1. Dodgson Condensation: birth of the ASM 2

1.2. Alternating Sign Matrices 5

1.3. $\boldsymbol{x}$-enumeration 6

1.4. Kuperberg 7

1.5. Square Ice Model 8

1.6. Partition Functions 12

1.7. Evaluations of the partition function 14

1.8. Beyond Kuperberg 15

2. The Yang-Baxter Equation 16

3. Partition Function of $\mathcal{C}_n^*$ Matrices 19

4. Subclasses of $\mathcal{C}_n^*$ Matrices 33

4.1. Class 1 Matrices 34

4.2. Class 2 Matrices 36

4.3. Class 2A and Class 2B Matrices 38

5. $M(n)$ Coefficient 39

6. $S(n; x_n, y_{n+1})$ Coefficient 47

7. Partition Function of $\mathcal{D}_n$ Matrices 52

References 61

TOWARDS ENUMERATIONS OF $\mathcal{C}_n^*$ AND $\mathcal{D}_n$ MATRICES

RAN JI

Abstract. Kuperberg [3] proved the conjecture on the number of alternating sign matrices (ASMs) of rank n using its partition function, found through its corresponding ice model. He went on to discover subclasses of ASMs with specific symmetry properties and their corresponding ice models, and was able to enumerate these matrices by making a connection between partition functions and x-enumerations. Using similar methods, Razumov and Stroganov [4] enumerated the class of half-turn symmetric alternating sign matrices of odd order by connecting its partition function to partition functions whose explicit formulas are known. We introduce two new classes of half-turn symmetric matrices, $\mathcal{C}_n^*$ and $\mathcal{D}_n$. For both the $\mathcal{C}_n^*$ and $\mathcal{D}_n$ matrices, we prove symmetry properties and recursive formulas for their partition functions. For the $\mathcal{C}_n^*$ matrices, we also find lead coefficients on specific lead terms in its partition function.

1. Introduction

1.1. **Dodgson Condensation: birth of the ASM.** Our problem began with the Dodgson condensation, a lesser known method for computing the determinants of square matrices. It is named after its inventor Charles Dodgson (better known by his pen name, Lewis Carroll, author of *Alice in Wonderland*). Let A be a $n \times n$ matrix with $a_{i,j}$ representing the entry in the ith row, jth column of A. The algorithm for the Dodgson condensation method is as follows:

(1) Using elementary row and column operations, rearrange A so that there are no zeros in the interior of A. Define *interior* of a matrix as elements not in the first and last rows and columns. More precisely, the interior of a $n \times n$ matrix is all entries with index (i, j) such that $i, j \neq 1, n$. See Figure 1 for an example of the interior of a square matrix. Note that this is only possible when the interior does not contain a row or column of 0's and that this can be done without changing the absolute value of the determinant.

(2) Create an $(n-1) \times (n-1)$ matrix B, where each entry is the determinant of one of A's 2×2 submatrices.

$$\begin{bmatrix} 1 & 1 & -5 & 6 & 2 \\ -3 & \boxed{4} & \boxed{2} & \boxed{0} & -1 \\ 5 & \boxed{-3} & \boxed{4} & \boxed{5} & 6 \\ -8 & \boxed{9} & \boxed{0} & \boxed{0} & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \qquad \begin{bmatrix} a & b & c \\ d & \boxed{e} & f \\ g & h & i \end{bmatrix}$$

FIGURE 1. The interiors of the matrices above are boxed.

(3) Repeat step 2 on B to obtain an $(n-2) \times (n-2)$ matrix C. Divide each term in C by the corresponding term in the interior of A (this is why there cannot be zero entries in the interior of A). In other words, let $c_{i,j} = \frac{b_{i,j}}{a_{i+1,j+1}}$.

(4) Repeat steps 1-3 by substituting B for A and C for B.

(5) The process ends when the result is a 1×1 matrix. The only entry in this matrix is the determinant of the original matrix A.

If we apply Dodgson condensation to the 3×3 matrix

$$B = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix},$$

we obtain the 2×2 matrix

$$\begin{bmatrix} \begin{vmatrix} a & b \\ d & e \end{vmatrix} & \begin{vmatrix} b & c \\ e & f \end{vmatrix} \\ \begin{vmatrix} d & e \\ g & h \end{vmatrix} & \begin{vmatrix} e & f \\ h & i \end{vmatrix} \end{bmatrix} = \begin{bmatrix} ae - bd & bf - ce \\ dh - eg & ei - fh \end{bmatrix}.$$

From this we can find the 1×1 matrix whose only entry is

$$[(ae^2i - aefh - bdei + bdfh) - (bdfh - befg - cdeh + ce^2g)]/e.$$

After collecting the terms, the expression becomes

$$(1)aei + (-1)afh + (-1)bdi + (1)bfg + (1)cdh + (1)ceg + (0)bde^{-1}fh. \tag{1.1}$$

Before we can dive further into the story, we must first introduce the concept of permutation matrices.

Definition 1.2 (Permutation matrix)**.**

Let s be an element of S_n, the symmetric group of rank n. The permutation matrix P_s resulting from s is the $n \times n$ matrix whose entries are all 0 except that in column j the entry $s(j)$ equals 1*.*

Colloquially, a permutation matrix of rank n is a $n \times n$ matrix with exactly one 1 in each row and column and 0 in all other entries. See Example 1.3 for an example of how one obtains a permutation matrix from an element of the symmetric group.

Example 1.3 (Permutation matrices)**.**

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$(1) \in S_3 \qquad (13)(24) \in S_4 \qquad (123)(45) \in S_5$$

In the polynomial in Equation (1.1), each of the first six terms correspond to one of the six permutation matrices of rank n. For each "variable" in a term, replace the position that it occupied in B with 1 and place 0's in all other positions:

$$\underset{aei}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \quad \underset{afh}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}} \quad \underset{bdi}{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \quad \underset{bfg}{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}} \quad \underset{cdh}{\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}} \quad \underset{ceg}{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}}$$

In addition to these terms, there is an extra term $(0)bde^{-1}fh$ that can be associated with the matrix with 1's in the positions of b, d, f and h and -1 in the position of e:

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

If we apply Dodgson condensation to the general $n \times n$ matrix, we would find that, in addition to the $n!$ monomials that make nonzero contributions to the determinant, there are additional monomials with coefficients of 0. Translating the monomials with nonzero coefficients into matrices gives us the complete set of permutation matrices of rank n. Translating every term (including the vanishing ones) into matrices gives the complete set of *alternating sign matrices* of rank n.

1.2. **Alternating Sign Matrices.** The alternating sign matrices were discovered by David Robbins and Howard Rumsey in their study of Dodgson condensation. Before formally defining them, we give the following useful definition.

Definition 1.4 (Sign-alternating)**.**
Let $\vec{a} = (a_1, a_2, a_3, \ldots, a_n)$ be a vector. Say that $\vec{a}$ is sign-alternating if

(1) $a_i \in \{0, 1, -1\}$ for all $1 \leq i \leq n$
(2) the nonzero entries alternate in sign, beginning with 1.

An equivalent definition is that when the vector is viewed as a sequence (indexed from left to right), the partial sum of its elements is always 0 or 1.

Example 1.5 (Sign-alternating vectors)**.**
The vectors below illustrate the idea of sign-alternating vectors:

- $\vec{a} = (1, -1, 1, -1, 0)$ *is sign-alternating*
- $\vec{b} = (0, -1, 1, -1, 1)$ *is not sign-alternating because its first nonzero element is -1, failing part (2) of the definition*
- $\vec{c} = (0, 1, -1, -1, 0)$ *is not sign-alternating because its nonzero elements do not alternate in sign, failing part (2) of the definition*

Definition 1.6 (Alternating sign matrix)**.**
Let A be an $n \times n$ matrix. A is called an alternating sign matrix (ASM) of rank n if

(1) every row and every column is sign-alternating
(2) $\sum_{i=1}^{n} a_{p,i} = \sum_{j=1}^{n} a_{j,q} = 1$ for all $1 \leq p, q \leq n$ (every row and every column sum to 1).

The set of all $n \times n$ ASMs is denoted by $\mathcal{A}_n$.

Example 1.7.
Examples of ASMs of different ranks:

$$
\begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad
\begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad
\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \tag{1.8}
$$

$$A \qquad\qquad B \qquad\qquad C$$

Robbins and Rumsey were later joined by William Mills, and together they set out to investigate $|\mathcal{A}_n|$. Using computer calculation, the three researchers found that the sequence $|\mathcal{A}_n|$ began with $1, 2, 7, 42, 429, 7436, 218348, 10850216, 911835460$. They were eventually able to arrive at a conjecture for $|\mathcal{A}_n|$, which was later proved by Doron Zeilberger [5].

Theorem 1.9 (ASM Conjecture [1])**.**

$$|\mathcal{A}_n| = \prod_{k=0}^{n-1} \frac{(3k+1)!}{(n+k)!} = \frac{1!4!7!\cdots(3n-2)!}{n!(n+1)!(n+2)!\cdots(2n-1)!}$$

Zeilberger's proof was announced in 1992 but his paper [5] was not published until 1996. His proof involved a bijection between ASMs and totally symmetric, self-complementary plane partitions. The paper was 84 pages in length and had a tree of nested lemmas with depth eight (the last level consisted of "sub^{7}lemmas"). Eighty-nine referees were needed to check the lemmas.

1.3. ***x*-enumeration.** In addition to finding $|\mathcal{A}_n|$, we can find the weighted enumerations of $\mathcal{A}_n$.

Definition 1.10 (Orbit)**.**
Let $M \in \mathcal{A}_n$. Define the orbit -1 to be the number of NS vertices in its corresponding ice state. This keeps track of the number of "distinct" -1's in the matrix. See Example 1.12.

Definition 1.11 (x-enumeration)**.**
Let $x \in \mathbb{Z}$ and let $\mathcal{M}$ be a subcollection of ASMs. If $M \in \mathcal{M}$ and if $\mu(M)$ is the number of orbits of -1 in M, assign to M the weight of $x^{\mu(M)}$. For the set $\mathcal{M}$ of ASMs, the x-enumeration of $\mathcal{M}$ is

$$|\mathcal{M}|_x = \sum_{M\in\mathcal{M}} x^{\mu(M)}.$$

Example 1.12.

Consider the three matrices in Equation (1.8).

- *Matrix A has one orbit of -1, $\mu(A) = 1$*
- *Matrix B has one orbit of -1, $\mu(B) = 1$*
- *Matrix C has two orbits of -1, $\mu(C) = 2$*

Note that when $x = 1$, $|\mathcal{M}|_1 = |\mathcal{M}|$ since each matrix is assigned a weight of 1.

1.4. **Kuperberg.** A few months after Zeilberger's paper was published, Greg Kuperberg found a much simpler proof using ideas from statistical mechanics.

In his quest to better understand ASMs, Kuperberg found that physicists had been studying ASMs in another form in their study of the structure of ice. As is often done in mathematics, physicists use a 2-dimensional lattice model to study the 3-dimensional lattice of water molecules in actual ice crystals. Figure 2 (taken from [1]) shows what is known as a patch of "square ice".

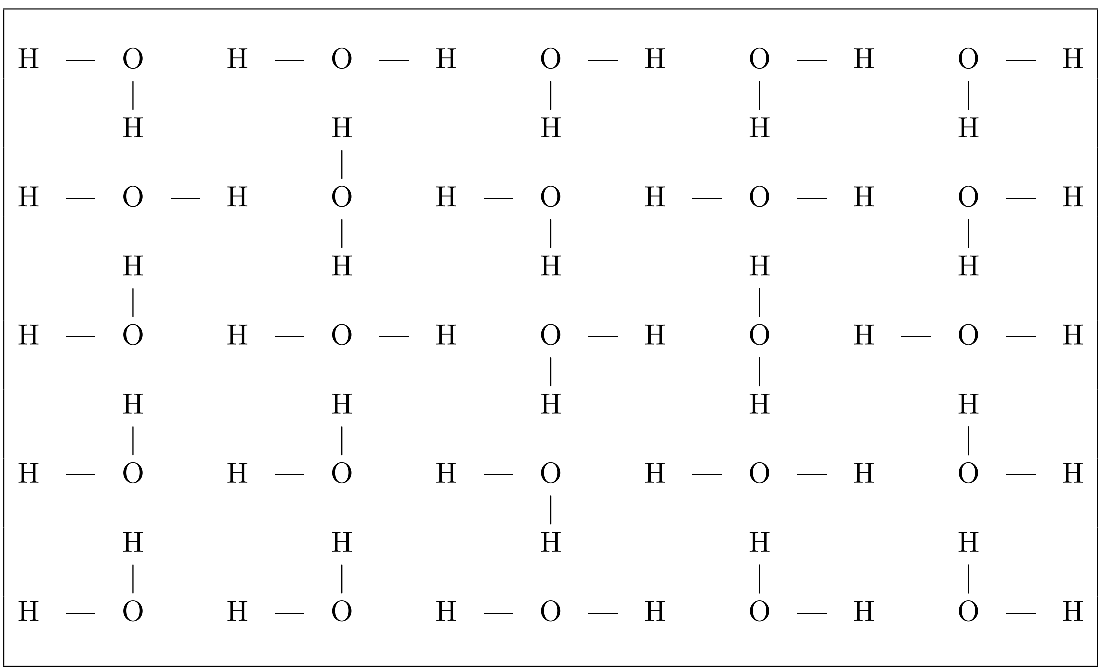


FIGURE 2. A patch of "square ice"

Any patch of "square ice" can be translate into an ASM by replacing horizontal molecules with 1, vertical molecules with -1, and angled molecules with 0. The patch shown in Figure 2 corresponds to the ASM

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}. \tag{1.13}$$

Such ice states were often represented by physicists as directed graphs on a square lattice with the restriction that each vertex has two edges pointing inward and two edges pointing outward. The translation from "square ice" to such graphs is done using the following method:

(1) Replace each oxygen atom (O) with a vertex

(2) Replace each hydrogen atom (H) with a directed edge pointing toward the oxygen atom to which it is attached

(3) If any vertex does not have four adjacent edges, replace missing edges with edges pointing away from the vertex.

Figure 3 shows the directed graph on a square lattice converted from Figure 2.

$$\begin{array}{ccccccccccc}
 & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
\rightarrow & \bullet & \rightarrow & \bullet & \leftarrow & \bullet & \leftarrow & \bullet & \leftarrow & \bullet & \leftarrow \\
 & \uparrow & & \downarrow & & \uparrow & & \uparrow & & \uparrow & \\
\rightarrow & \bullet & \leftarrow & \bullet & \rightarrow & \bullet & \rightarrow & \bullet & \leftarrow & \bullet & \leftarrow \\
 & \downarrow & & \uparrow & & \uparrow & & \downarrow & & \uparrow & \\
\rightarrow & \bullet & \rightarrow & \bullet & \leftarrow & \bullet & \leftarrow & \bullet & \rightarrow & \bullet & \leftarrow \\
 & \downarrow & & \downarrow & & \uparrow & & \uparrow & & \downarrow & \\
\rightarrow & \bullet & \rightarrow & \bullet & \rightarrow & \bullet & \rightarrow & \bullet & \leftarrow & \bullet & \leftarrow \\
 & \downarrow & & \downarrow & & \uparrow & & \downarrow & & \downarrow & \\
\rightarrow & \bullet & \rightarrow & \bullet & \rightarrow & \bullet & \leftarrow & \bullet & \leftarrow & \bullet & \leftarrow \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow &
\end{array}$$

FIGURE 3. Figure 2 converted into a directed graph on a square lattice

Notice that the boundary of the graph in Figure 3 is such that all of the arrows along the left and right side point inward while the arrows along the top and bottom point outward. This boundary condition is called the *domain wall boundary condition* for the square ice model. States satisfying this boundary condition are the square-ice states that are equivalent to ASMs.

1.5. **Square Ice Model.** In the section below we hope to make more precise the concepts introduced in the section above.

A *directed graph* $\mathcal{G}$ is a collection of vertices $V(\mathcal{G})$ together with a collection of distinct edges $E(\mathcal{G})$, where each edge can be thought of as an ordered pair of vertices. A directed edge is represented by an arrow pointing from the first vertex to the second. If a vertex is connected to only one edge, we call it *univalent*. A vertex that is connected to four edges is called *tetravalent*. In Figure 5, each vertex on the boundary (outer rim) of the graph is univalent and all other vertices are tetravalent.

Definition 1.14 (Ice state, square ice)**.**
Let $\mathcal{G}$ be a directed graph such that each internal vertex is tetravalent and each boundary vertex is univalent. An ice state (also called a six-vertex state) of $\mathcal{G}$ is an orientation of the edges such that exactly two edges enter each tetravalent vertex. If $\mathcal{G}$ is also a square grid, then the set of ice states is called square ice*.*

If the orientations of the edges connected to the univalent vertices of $\mathcal{G}$ are fixed, we call this the *boundary conditions*. The boundary condition shown in Figure 5 is called *domain wall boundary*, the boundary condition associated with "square ice." If we replace the tetravalent vertices in a square ice with domain wall boundary with numbers as shown in Figure 4, a square matrix is formed. This method of replacement is equivalent to the method used for replacing Figure 2 with the matrix in Equation (1.13).

Proposition 1.15 ([2])**.**
The set of $n \times n$ square ice with domain wall boundary is in bijection with $\mathcal{A}_n$.

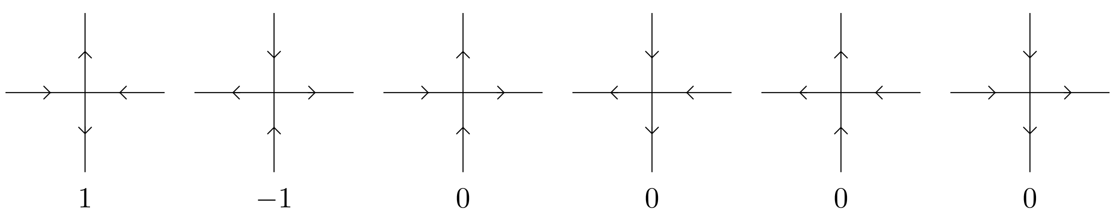


FIGURE 4. Replacing square ice with alternating-sign entries

Although Kuperberg first used square ice with domain wall boundary to prove the ASM Conjecture, he noticed that subcollections of $\mathcal{A}_n$ with certain symmetries could be put in bijection with certain "bent" ice models. For example, Kuperberg extensively studied matrices with *half-turn symmetry*.

Definition 1.16 (Half-turn symmetry)**.**
Let A be an $n \times m$ matrix. Let $a_{i,j}$ denote the entry in row i and column j of A. Then A is

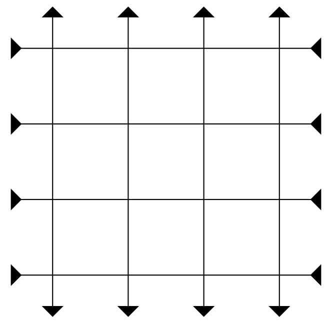


FIGURE 5. 4×4 square ice with domain wall boundary.

half-turn symmetric *(HTS) if for* $1 \leq i \leq n$ *and* $1 \leq j \leq m$*, it is the case that* $a_{i,j} = a_{n+1-i,m+1-j}$*. More intuitively,* A *is half-turn symmetric if rotating it by* $180°$ *gives back* A*.*

See Figure 6 for examples of half-turn symmetric alternating-sign matrices.

$$\begin{bmatrix} 0 & 0 & 0 & \mathbf{1} \\ 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 \\ \mathbf{1} & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & \mathbf{1} & \mathbf{-1} & \mathbf{1} \\ 0 & \mathbf{1} & \mathbf{-1} & \mathbf{1} & 0 \\ \mathbf{1} & \mathbf{-1} & \mathbf{1} & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 & 0 \end{bmatrix}$$

FIGURE 6. A 4×4 and 5×5 half-turn symmetric ASM. Feel free to verify the half-turn symmetry by rotating the paper 180°. The nonzero entries have been bolded for ease of reading.

For an $n \times n$ square matrix, the formal definition of half-turn symmetric dictates that entry $a_{i,j}$ is equal to entry $a_{n+1-i,n+1-j}$ for all i and j. The nature of half-turn symmetry makes it possible to partition the vertices into two sets such that for each entry $a_{i,j}$, its HTS "complement" entry $a_{n+1-i,n+1-j}$ is in the other set. For a $2n \times 2n$ HTSASM, an example would be to place entries in rows 1 through n in one set and entries in rows $n+1$ through $2n$ in the other. Alternatively, one could place entries in columns 1 through n in one set, and entries in columns $n+1$ to $2n$ in the other. We will frequently use this method. If the entries are partitioned in this way, then knowing the values of the entries in one of the partitioned set is enough for us to reconstitute the entire matrix. Thus, a corresponding ice model needs only half the number vertices.

Definition 1.17 ($\mathcal{B}_n$)**.**

The set of HTSASM of dimension $2n \times 2n$ *is denoted* $\mathcal{B}_n$*. A matrix* $B \in \mathcal{B}_n$ *is called an even HTSASM of rank* n*.*

The matrix on the left of Figure 6 is an example of an even HTSASM of rank 2, i.e. an element of $\mathcal{B}_2$.

Kuperberg showed in [3] that the set of $2n \times n$ ice states with even HTSASM boundary condition (shown in Figure 7) is in bijection with the elements of $\mathcal{B}_n$. We will not repeat his proof but illustrate the bijection in Figure 8. We begin, on the left, with an ice state satisfying the even HTSASM boundary condition. Translating the vertices into numbers using the chart given in Figure 4, we arrive at the pseudo-matrix in the middle. Finally, we are able to reconstitute the rest of the matrix using half-turn symmetry.

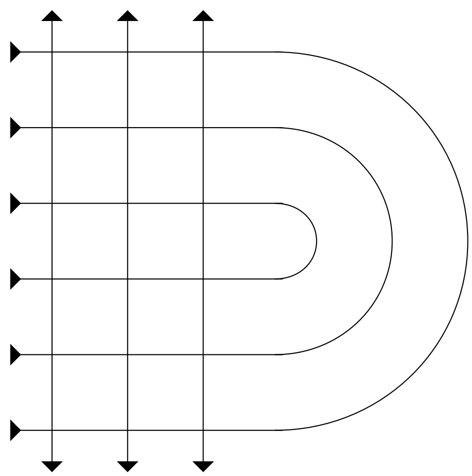
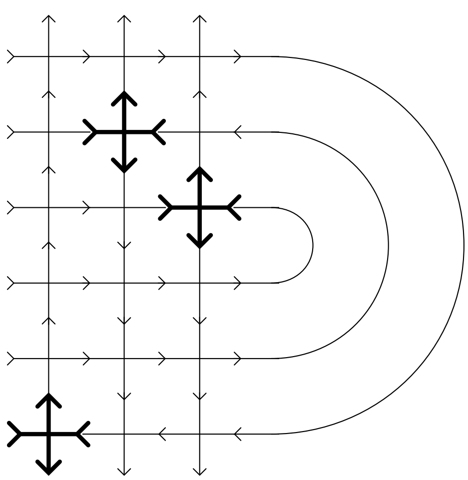


FIGURE 7. Even HTS boundary condition when $n = 3$



$$\Leftrightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \Leftrightarrow \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \mathbf{1} \\ 0 & \mathbf{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} & 0 \\ \mathbf{1} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

FIGURE 8. Showing the bijection between square ice with even HTSASM boundary condition and elements of $\mathcal{B}_n$. The vertices corresponding to nonzero entries and nonzero matrix entries have been bolded for ease of reading.

1.6. **Partition Functions.** Physicists studying the structure of square ice were interested in weighted sums taken over all possible configurations of a given size and satisfying given boundary conditions, as each such sum was a single entity that encoded the information of all possible configurations of given size and boundary condition. This weighted sum, roughly speaking, is the partition function of the given boundary conditions and size.

For the remainder of the paper, we will use the following notation (to be consistent with the notation used by Kuperberg, Razumov and Stroganov):

$$\begin{aligned} \bar{x} &= x^{-1} \\ \sigma(x) &= x - \bar{x}. \end{aligned}$$

For each line in the graph, we can assign to it a spectral parameter. The horizontal lines are now associated with spectral parameters x_i while the vertical lines are associated with spectral parameters y_j (See Figure 9). A vertex at the intersection of the lines with spectral parameters x_i and y_j is given the spectral parameter $x_i\bar{y}_j$. For example, the lower right vertex in Figure 9 has spectral parameter $x_4\bar{y}_4$.

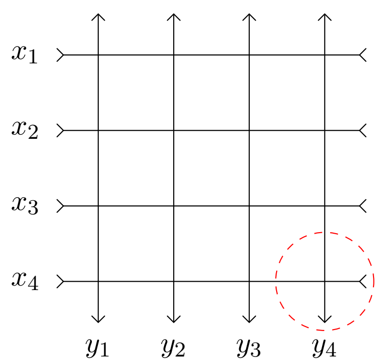
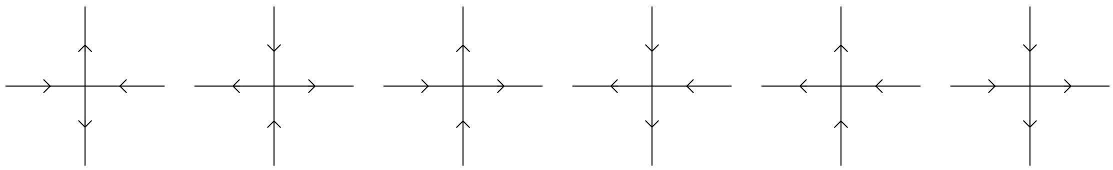


FIGURE 9. Domain wall boundary labeled with spectral parameters. The circled vertex has spectral parameter $x_4\bar{y}_4$.

Now that each vertex is associated with a spectral parameter, we can assign weights to the vertices depending on the configuration of the vertex; these weights are referred to as *Boltzmann weights*. Each vertex configuration is named using the compass direction corresponding to the two inward arrows. Figure 10 assumes that the vertex has general spectral parameter $x_i\bar{y}_j$ and gives the names and weights of each vertex configuration. Parameter a is common for all vertices.

The weight assigned to a particular square ice is simply the product of the vertex weights of all its vertices.

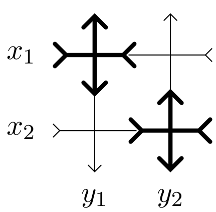


EW	NS	SW	NE	SE	NW
$\sigma\left(a^2\right)$	$\sigma\left(a^2\right)$	$\sigma(ax_i\bar{y}_j)$	$\sigma(ax_i\bar{y}_j)$	$\sigma(a\bar{x}_iy_j)$	$\sigma(a\bar{x}_iy_j)$
1	-1	0	0	0	0

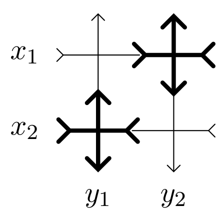
FIGURE 10. Name, weight, and equivalent matrix entry of each vertex configuration

Definition 1.18 (Weight of a Matrix)**.**
Suppose M is an ASM represented by an ice state $\rho(M)$. Then then weight of M, denoted $wt(M)$, is the product of the vertex weights from Figure 10, taken across all vertices of $\rho(M)$. We also call this polynomial the weight of $\rho(M)$.

Example 1.19 (Weight of a matrix)**.**
There are only two 2×2 square ice satisfying the boundary wall condition. Recall that they are in bijection with $\mathcal{A}_2$. Below we show the two square ice and their corresponding ASMs. We will name the ice states α and β and the matrices A and B, from left to right.



$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$



$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Using the weights given in Figure 10, we calculate the weights of α and β:

$$wt(\alpha) = \left(\sigma\left(a^2\right)\right)^2 \sigma(a\bar{x}_1y_2)\sigma(a\bar{x}_2y_1), wt(\beta) = \left(\sigma\left(a^2\right)\right)^2 \sigma(ax_1\bar{y}_1)\sigma(ax_2\bar{y}_2).$$

Definition 1.20 (Partition function)**.**
Let $\mathcal{M}$ be a class of ASMs, and for each $M \in \mathcal{M}$, let $\rho(M)$ be the ice state which represents M. Then the partition function for $\mathcal{M}$ is defined as

$$Z_{\mathcal{M}}(\vec{x}, \vec{y}) = \sum_{M \in \mathcal{M}} wt\left(M\right) = \sum_{M \in \mathcal{M}} wt\left(\rho\left(M\right)\right).$$

The partition function of 2×2 square ice satisfying wall boundary condition is the sum of the weights of α and β (from Example 1.19) since they are the only square ice satisfying the wall boundary condition. The partition function of ASMs of rank 2 is equal to:

$$Z_{\mathcal{A}_2}\left((x_1, x_2), (y_1, y_2)\right) = \left(\sigma\left(a^2\right)\right)^2 \left[\sigma(a\bar{x}_1 y_2)\sigma(a\bar{x}_2 y_1) + \sigma(ax_1\bar{y}_1)\sigma(ax_2\bar{y}_2)\right].$$

Note that although partition functions are calculated using an ice model, they are named using by the set of matrices that the specific ice model represent.

Referring back to Example 1.19, we see that both A and B have even rank and are HTS. Therefore, they are elements of $\mathcal{B}_1$. Since $\mathcal{B}_1 \subset \mathcal{A}_2$ and $\mathcal{A}_2 = \{A, B\} \subset \mathcal{B}_1$, we conclude that $\mathcal{B}_1 = \mathcal{A}_2$. As elements of $\mathcal{B}_1$, A and B have corresponding "bent" ice states satisfying the HTSASM boundary condition, shown in Figure 11.

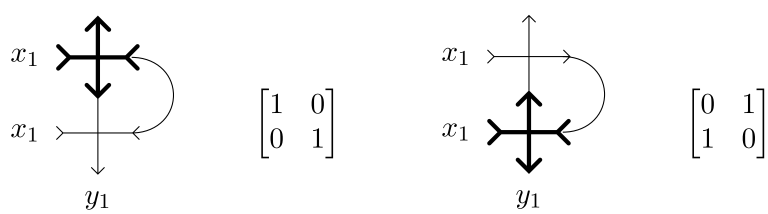


FIGURE 11. All 2×1 "bent" ice satisfying the even HTSASM boundary condition.

Now that we have the "bent" ice representation of $\mathcal{B}_1$, we can find its partition function. Summing over the weights of A and B, we find that the partition function is equal to:

$$Z_{\mathcal{B}_1}\left((x_1), (y_1)\right) = \sigma(a^2)\left[\sigma\left(a\bar{x}_1 y_1\right)\sigma\left(ax_1\bar{y}_1\right)\right].$$

It's interesting to note that although $\mathcal{A}_2$ and $\mathcal{B}_1$ are the same set, their partition functions are not equal because they correspond to different ice states with different boundary conditions.

1.7. **Evaluations of the partition function.** Kuperberg recognized that evaluating the partition function at particular values for the x_i parameters, y_i parameters and a parameter gives the enumeration for the corresponding class of ASMs. It was using this method that Kuperberg proved the ASM Conjecture (details can be found in [3]).

Theorem 1.21 ([3]).
When $a = e^{\frac{2\pi\imath}{3}}$,

$$|\mathcal{A}_n|_1 = |\mathcal{A}_n| = \frac{Z_{\mathcal{A}_n}(\vec{1},\vec{1})}{\sigma(a)^{n^2-n}\sigma\left(a^2\right)^n},$$
$$|\mathcal{B}_n|_1 = |\mathcal{B}_n| = \frac{Z_{\mathcal{B}_n}(\vec{1},\vec{1})}{\sigma(a)^{2n^2-n}\sigma\left(a^2\right)^n}.$$

Kuperberg's big contribution was that his method was able to give not only the 1-enumeration for classes of ASMs, but more refined enumerations as well. He found that evaluating the partition function at other special values gives counts of x-enumerations of classes of ASMs. Theorem 1.22 highlights some of his results.

Theorem 1.22 ([3]).

$$|\mathcal{A}_n|_2 = \frac{Z_{\mathcal{A}_n}(\vec{1},\vec{1})}{\sigma(a)^{n^2-n}\sigma\left(a^2\right)^n} \quad \textit{when } a = \sqrt{-1}$$
$$|\mathcal{B}_n|_2 = \frac{Z_{\mathcal{B}_n}(\vec{1},\vec{1})}{\sigma(a)^{2n^2-n}\sigma\left(a^2\right)^n} \quad \textit{when } a = \sqrt{-1}$$
$$|\mathcal{A}_n|_3 = \frac{Z_{\mathcal{A}_n}(\vec{1},\vec{1})}{\sigma(a)^{n^2-n}\sigma\left(a^2\right)^n} \quad \textit{when } a = e^{\frac{2\pi\imath}{6}}$$

1.8. **Beyond Kuperberg.** After proving the ASM Conjecture, Kuperberg used ice models to enumerate subclasses of ASMs with particular symmetry properties in [3]. In addition to the enumeration of even HTSASMs, he gave counts to quarter-turn-symmetric ASMs (invariant under 90° rotations), vertically symmetric ASMs (symmetric with respect to the vertical "bisector" of the matrix), and many others. Alexander Razumov and Yuri Stroganov used many of Kuperberg's ideas to find enumerations of odd HTSASMs in [4]. Their article was used as the primary guide for this thesis.

Definition 1.23 ($\mathcal{C}_n$).
The set of HTSASM of dimension $(2n+1)\times(2n+1)$ *is denoted* $\mathcal{C}_n$. *A matrix* $C \in \mathcal{C}_n$ *is called an odd HTSASM of rank* n.

In 2012, Ben Brubaker and Andrew Schultz found a connection between ice models and Weyl denominator formulas, which are connected to Lie groups of classical types. Their discovery led to the study of two new types of ASMs, which they called $\mathcal{C}_n^*$ and $\mathcal{D}_n$, and their corresponding ice models. They are not alternating-sign matrices in the strictest sense since they are not square matrices. However, they share some common characteristics with ASMs. All entries are 1, 0, or -1 and each row and column is sign-alternating. Both types of matrices are half-turn symmetric. In Section 3 and beyond, we detail our study of the $\mathcal{C}_n^*$ and $\mathcal{D}_n$ matrices.

The aim of this thesis is to find explicit formulas for $Z_{\mathcal{C}_n^*}(\vec{x},\vec{y})$ and $Z_{\mathcal{D}_n}(\vec{x},\vec{y})$ toward the ultimate goal of enumerating the two classes of matrices. Our strategy for computing the two partition functions is to study certain symmetry properties and lead coefficients on these two polynomials as a kind of "signature." If we had more time, we would have then found other polynomials with the same "signatures" and concluded that they were our desired partition functions based on a result found by Razumov and Stroganov in [4]. From there, we would have used the x-enumeration method to enumerate the matrices, as Kuperberg did.

2. The Yang-Baxter Equation

To determine $Z_{\mathcal{C}_n}(\vec{x},\vec{y})$, Razumov and Stroganov proved that if two functions have certain symmetry and recursive properties and have identical lead coefficient on a specific term (both ideas are discussed in later sections), then the two functions must be equal. See Theorem 3.22 for the equivalent result for $\mathcal{C}_n^*$. For each of $\mathcal{C}_n^*$ and $\mathcal{D}_n$ matrices, our overall strategy is to prove that if a function has the desirable properties, then it must be the partition function. Since our ultimate goal is to enumerate $\mathcal{C}_n^*$ and $\mathcal{D}_n$ matrices, the hope is that we can then use the x-enumeration method that Kuperberg employed to find their enumerations.

In order to prove one of the symmetry properties for the partition function of $\mathcal{C}_n^*$ and $\mathcal{D}_n$, we need the Yang-Baxter Equation. This equation was first introduced in the field of statistical mechanics and takes its name from independent work of C. N. Yang from 1968 and R. J. Baxter from 1971. The Yang-Baxter Equation appears in a variety of contexts, including electric networks, knot theory, and braid groups. For us, this equation means that passing a "twist" through an ice model with fixed boundary condition does not change the sum weight of over all configurations satisfying said boundary condition.

Consider the "twist" diagrams in Figure 12. In them, $\{\alpha, \beta, \gamma, \delta, \epsilon, \phi\}$ represent the six exterior edges at the six boundary positions, and $\{A, B, C, D, E, F\}$ represent unoriented edges in the diagrams.

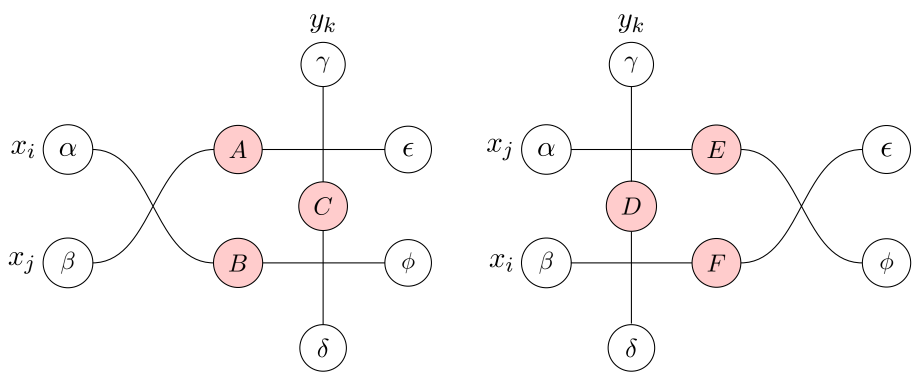


FIGURE 12. "Passing through" the twist

The second diagram can be obtained from the first by "passing through" the twist. Consider both twist diagrams as possible components of an ice model. In filling in the internal edges of the two diagrams (labeled A, B, C, D, E, and F), we must ensure that there are exactly two arrows going into each vertex.

As we did with ice models, we can assign weights to each vertex based on its configuration. Our choice of weights come from an appropriate translation of the weighting scheme from Figure 10. Rotating the twists by 45° counterclockwise will give the corresponding vertex configuration. This means that the line going from the upper left corner to the lower right corner acts as our usual 'x' spectral parameter, while the line going from the lower left corner to the upper right corner acts as the 'y' parameter. Thus, if the first line has spectral parameter x_i and the second has spectral parameter x_j, the vertex at their intersection would have spectral parameter $x_i\bar{x}_j$. There is an additional a parameter, to distinguish the twisted vertex from the non-twisted ones. The weighting system used for the twists is given in Figure 13.

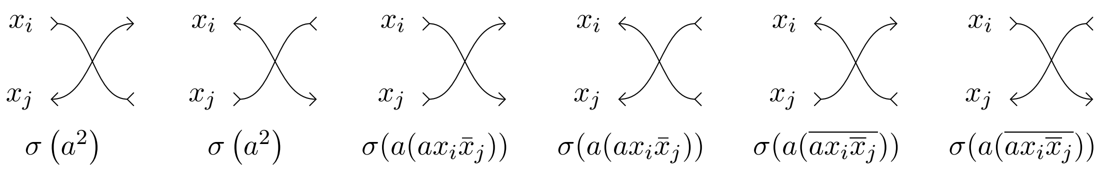
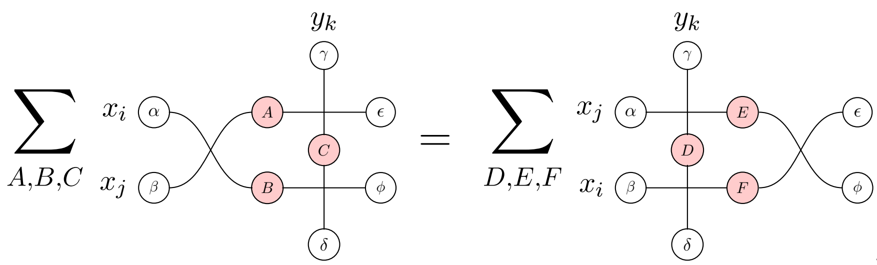


FIGURE 13. Weighting system twist vertices. Each has spectral parameter $x_i\bar{x}_j$. When simplified, the third and fourth configurations have weight $\sigma\left(a^2x_i\bar{x}_j\right)$ and the fifth and sixth configurations have weight $\sigma\left(\bar{x}_ix_j\right)$.

Theorem 2.1 (Yang-Baxter Equation).

If the three lines are assigned spectral parameters $\{x_i, x_j, y_k\}$, and if $\{\alpha, \beta, \gamma, \delta, \epsilon, \phi\}$ are fixed, then



In words, summing over the weight of all possible ways of filling out the twist diagram on the left will give the same expression as summing over the weight of all possible ways of filling out the twist diagram on the left.

Kuperberg proved the Yang-Baxter Equation in this context in [3]. We will illustrate the theorem with an example. Let α, δ, ϕ be in, and let β, γ, ϵ be out. Then the two twist diagrams become the diagrams shown in Figure 14.

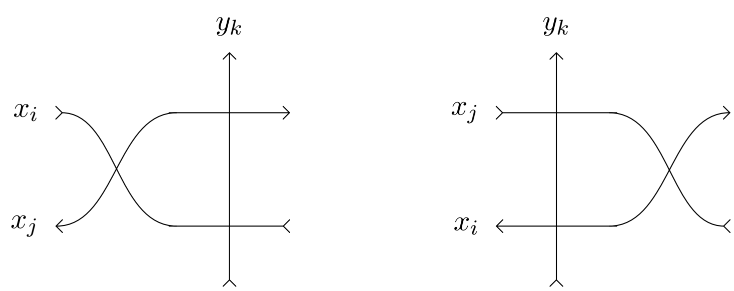


FIGURE 14. Twist diagrams for fixed $\{\alpha, \beta, \gamma, \delta, \epsilon, \phi\}$

For the twist diagram on the left, vertex y having two arrows leaving and vertex x having two arrows coming in completely dictates the rest of the arrows; there is only one way to fill out this diagram. On the right, there are two ways to fill out the diagram with arrows. Figure 15 shows these three diagrams and the names we've given them. The weights of the three twist diagrams shown in Figure 15 are given below:

- $wt(\omega) = \sigma\left(a^2\right)\sigma\left(ax_i\bar{y}_k\right)\sigma\left(a\bar{x}_j y_k\right)$
- $wt(\tau) = \sigma\left(ax_j\bar{y}_k\right)\sigma\left(a\bar{x}_i y_k\right)\sigma\left(a^2\right)$
- $wt(\pi) = \sigma\left(a^2\right)\sigma\left(a^2\right)\sigma\left(\bar{x}_j x_i\right)$

The Yang-Baxter Equation tells us that $wt(\omega) = wt(\tau) + wt(\pi)$, which has been verified by the author.

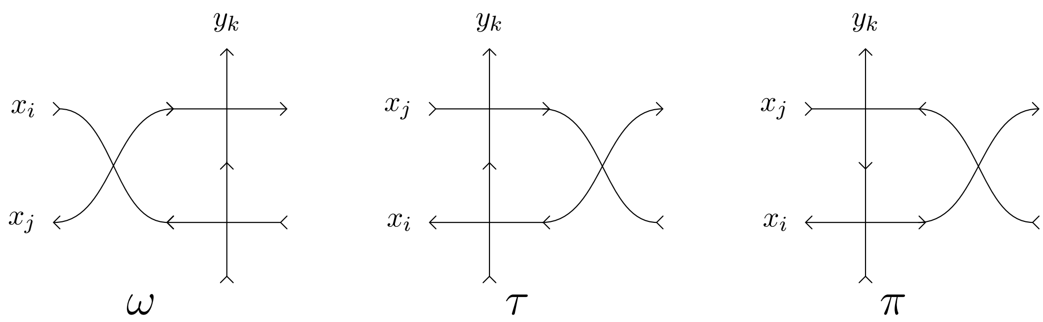


Figure 15

Note that although we have only shown the Yang-Baxter Equation for twisting of two horizontal lines, the equivalent is true for twisting of two vertical lines.

3. Partition Function of $\mathcal{C}_n^*$ Matrices

The $\mathcal{C}_n^*$ (pronounced "C alt") matrices are similar to HTSASMs in that they have half-turn symmetry, rows and columns are sign-alternating, and almost every row and column sum to 1. The difference is that $\mathcal{C}_n^*$ matrices are not square and the middle column has special restrictions. The formal definition is given below.

Definition 3.1 ($\mathcal{C}_n^*$ Matrices)**.**
Let $\mathcal{C}_n^*$ be the set of all $2n \times (2n+1)$ matrices $A = (a_{i,j})_{1\leq i\leq 2n, 1\leq j\leq 2n+1}$ satisfying the following conditions:

(1) For all $1 \leq p \leq 2n$, row p is sign-alternating with $\sum_{j=1}^{2n+1} a_{p,j} = 1$.

(2) For all $1 \leq q \leq 2n+1$, except for $q = n+1$, column q is sign-alternating with $\sum_{i=1}^{2n} a_{i,q} = 1$.

(3) The vector $(a_{1,n+1}, \ldots, a_{n,n+1})$ is sign-alternating and $\sum_{i=1}^{n} a_{i,n+1} = 0$.

(4) $a_{i,j} = a_{2n+1-i,2n+2-j}$.

Condition (1) and (2) tell us that all rows and all columns (except for the middle column) are sign-alternating and that their entries sum to 1. This forces the rows and columns with nonzero entries to begin and end with 1. Condition (3) says that the top half of the middle column is sign-alternating and must sum to 0. Then, if the top half of the middle column has nonzero entries, they will begin with 1 and end with -1. Condition (4) is the half-turn symmetric condition.

As is the case with elements of $\mathcal{A}_n$ and $\mathcal{B}_n$, there exists a specific boundary condition on an ice state that correspond to the set of $\mathcal{C}_n^*$ matrices. See Figure 17 for an illustration of this bijection.

Theorem 3.2 (Bijection between $\mathcal{C}_n^*$ and ice states)**.**
There exists a bijection between $\mathcal{C}_n^*$ and ice states satisfying the $\mathcal{C}_n^*$ boundary condition, shown in Figure 16. Let Γ_n^* be the set of ice states corresponding to the set of $\mathcal{C}_n^*$ matrices.

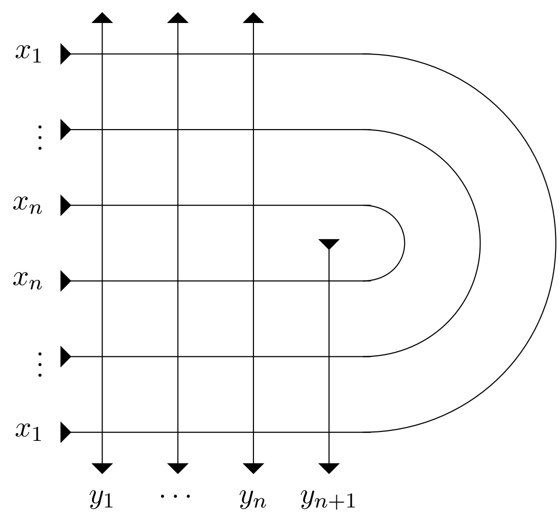
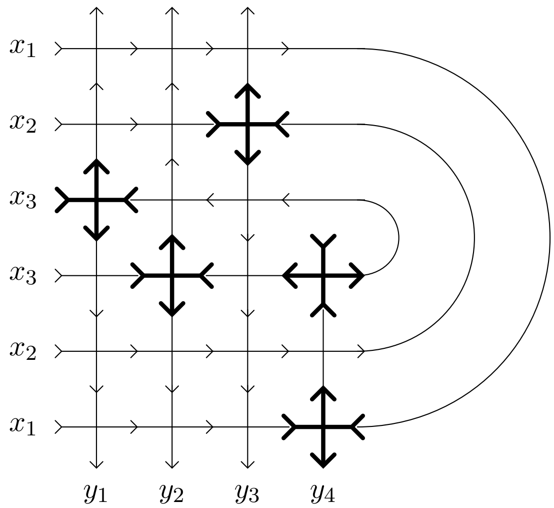


FIGURE 16. $\mathcal{C}_n^*$ boundary condition

Similar to the ice states in bijection with $\mathcal{B}_n$, this ice model only "records" half of the entries in a $2n \times (2n+1)$ matrix, but the half-turn symmetric nature of the matrices tells us precisely what the other entries are. Translating the vertices in an ice model with the $\mathcal{C}_n^*$ boundary condition into numbers using the chart given in Figure 4, we get the lower left half of a $\mathcal{C}_n^*$ matrix (the pseudo-matrix in the middle of Figure 17).

Recall that if the lines in an ice state with boundary condition are given spectral parameters, then we can calculate the weight of the ice state. Define "states of a boundary condition" as ways of filling out the graph such that each vertex has two inward and two outward edges. Summing over all possible states of a set dimension satisfying that boundary condition gives the partition



$$\iff \begin{pmatrix} 0 & 0 & 0 & \\ 0 & 0 & 1 & \\ 1 & 0 & 0 & \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \iff \begin{bmatrix} 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 \\ \mathbf{1} & 0 & 0 & \mathbf{-1} & 0 & \mathbf{1} & 0 \\ 0 & \mathbf{1} & 0 & \mathbf{-1} & 0 & 0 & \mathbf{1} \\ 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 \end{bmatrix}$$

FIGURE 17. Illustration of the bijection between the $\mathcal{C}_n^*$ boundary condition and $\mathcal{C}_n^*$ matrices

function. When we gave the $\mathcal{C}_n^*$ boundary condition in Figure 16, we labeled the lines with the spectral parameters that they have been assigned. For a $\mathcal{C}_n^*$ ice state, the spectral parameters for its lines are $\{x_1, x_2, \ldots, x_n, y_1, y_2 \ldots, y_n, y_{n+1}\}$.

If we sum over the weights all states of the $\mathcal{C}_n^*$ boundary condition, what we get back is the partition function associated with $\mathcal{C}_n^*$ matrices.

Definition 3.3 ($Z_{\mathcal{C}_n^*}(\vec{x}, \vec{y})$)**.**
Let $Z_{\mathcal{C}_n^*}(\vec{x}, \vec{y})$ denote the partition function of $\mathcal{C}_n^*$ matrices. It is equal to the sum over the weights of all elements in Γ_n^*, i.e.

$$Z_{\mathcal{C}_n^*}(\vec{x}, \vec{y}) = \sum_{\alpha \in \Gamma_n^*} wt\,(\alpha)\,.$$

Note that $Z_{\mathcal{C}_n^*}(\vec{x}, \vec{y})$ will have negative powers of the variables x_i's and y_i's. It would be better if we had a scaled version of the partition function with only non-negative powers of the x_i's and y_i's to make the algebra cleaner. In order to do this, we need to know the maximum possible degree for each of $\bar{x}_i$ and $\bar{y}_j$ in a $\mathcal{C}_n^*$ matrix.

Let A be a $\mathcal{C}_n^*$ matrix and let $\alpha \in \Gamma_n^*$ be the corresponding ice model. Because each row in A must sum to 1, we know that every row has at least one 1. Referring back to Figure 10, we see that a 1 or -1 entry in A corresponds to a factor of $\sigma\left(a^2\right) = a^2 - \bar{a}^2$ in the weight of the α. Only vertex configurations corresponding to a 0 entry contribute factors of x_i and y_i. Let us consider some general x_i. Because of the way the horizontal lines are labeled, only rows i and $2n+1-i$ can contribute factors of x_i. Since we are trying to maximize the power of $\bar{x}_i$, we want to maximize the number of 0's in row i and row $2n+1-i$, or equivalently, minimize the number of nonzero entries. The way to do this is to have exactly one 1 in each row. Because A is half-turn symmetric, having a 1 in the left half of row i means that there is a 1 in the right half of row $2n+1-i$, and vice versa. Since only the left half of the entries in A translate into vertices in α, only one of the 1's affect the weight of α. Then, in rows i and $2n+1-i$ of α, there is at least one vertex that does not contribute factors of $\bar{x}_i$. There are a total of $n+(n+1)=2n+1$ vertices in rows i and $2n+1-i$. Thus, the maximum possible power for $\bar{x}_i$ is $(2n+1)-1=2n$. The maximum possible powers for the $\bar{y}_i$'s can be found using similar reasoning.

For a $\mathcal{C}_n^*$ matrix, the maximum degree possible for $\bar{x}_i$ is $2n$, for $\bar{y}_i$ it is $2n-1$ $(1 \leq i \leq n)$, and for y_{n+1} it is n. Thus, we choose $y_{n+1}^n \left[\prod_{i=1}^n x_i^{2n} y_i^{2n-1}\right]$ as the scale factor.

Definition 3.4 ($\tilde{Z}_{\mathcal{C}_n^*}(\vec{x},\vec{y})$)**.**
Define a modified version of the partition function as

$$\tilde{Z}_{\mathcal{C}_n^*}(\vec{x},\vec{y}) = y_{n+1}^n \left[\prod_{i=1}^n x_i^{2n} y_i^{2n-1}\right] Z_{\mathcal{C}_n^*}(\vec{x},\vec{y}).$$

At the beginning of Section 2, we mentioned that there are certain symmetry and recursive properties used in a lemma (Lemma 17 in [4]) by Razumov and Stroganov in their investigation of $\tilde{Z}_{\mathcal{C}_n}(\vec{x},\vec{y})$. The following lemmas prove that $\tilde{Z}_{\mathcal{C}_n^*}(\vec{x},\vec{y})$ has these properties.

Lemma 3.5 (Homogeneity of $\tilde{Z}_{\mathcal{C}_n^*}(\vec{x},\vec{y})$)**.**
The modified function $\tilde{Z}_{\mathcal{C}_n^*}(\vec{x},\vec{y})$ is a homogeneous polynomial in the variables x_i and y_i of total degree $4n^2$. For each fixed $i \in \{1,\ldots,n\}$, the modified partition function is a polynomial in x_i^2 of degree $2n$ and a polynomial in y_i^2 of degree $2n-1$. It is a polynomial in y_{n+1}^2 of degree n.

Proof. Let us first consider $Z_{\mathcal{C}_n^*}(\vec{x},\vec{y})$. Let A be a $\mathcal{C}_n^*$ matrix. Let α be the ice state in Γ_n^* corresponding to A. Consulting Figure 10, we see that each vertex in α contributes either $\sigma\left(a^2\right) = a^2 - \bar{a}^2$, $\sigma(a x_i \bar{y}_j) = a\bar{x}_i y_j - \bar{a}\bar{x}_i y_j$, or $\sigma(a\bar{x}_i y_j) = a\bar{x}_i y_j - \bar{a} x_i \bar{y}_j$ to the weight of α. Notice that every term (the terms being $a^2, -\bar{a}^2, a x_i \bar{y}_j, \bar{a}\bar{x}_i y_j$, etc.) in every factor has degree 0 with respect to $\{x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_{n+1}\}$. The weight of α is the product of the weights of its vertices, so it must have the form

$$wt(\alpha) = (a^2 - \bar{a}^2)^{b_\alpha} (a\bar{x}_i y_j - \bar{a}\bar{x}_i y_j)^{c_\alpha} (a\bar{x}_i y_j - \bar{a} x_i \bar{y}_j)^{d_\alpha}.$$

Then the weight of α is a sum of products of terms with degree zero. Thus each term in $wt(\alpha)$ has degree zero. Since $Z_{\mathcal{C}_n^*}(\vec{x}, \vec{y})$ is a sum over the weights of all $\alpha \in \Gamma_n^*$, we know that it has the form

$$\sum_{\alpha \in \Gamma_n^*} wt(\alpha) = \sum_{\alpha \in \Gamma_n^*} (a^2 - \bar{a}^2)^{b_\alpha} (a\bar{x}_i y_j - \bar{a}\bar{x}_i y_j)^{c_\alpha} (a\bar{x}_i y_j - \bar{a} x_i \bar{y}_j)^{d_\alpha}.$$

Then each term in $Z_{\mathcal{C}_n^*}(\vec{x}, \vec{y})$ has degree 0.

We know that $\tilde{Z}_{\mathcal{C}_n^*}(\vec{x}, \vec{y})$ is equal to $y_{n+1}^n \left[\prod_{i=1}^n x_i^{2n} y_i^{2n-1}\right]$ times $Z_{\mathcal{C}_n^*}(\vec{x}, \vec{y})$. We've proven above that each term in $Z_{\mathcal{C}_n^*}(\vec{x}, \vec{y})$ has degree zero, thus each term in $\tilde{Z}_{\mathcal{C}_n^*}(\vec{x}, \vec{y})$ has degree equal to the degree of $y_{n+1}^n \left[\prod_{i=1}^n x_i^{2n} y_i^{2n-1}\right]$. The degree of $y_{n+1}^n \left[\prod_{i=1}^n x_i^{2n} y_i^{2n-1}\right]$ is

$$n + (2n)(n) + (2n-1)(n) = n + 2n^2 + 2n^2 - n = 4n^2.$$

By the definition of homogeneous, we conclude that $\tilde{Z}_{\mathcal{C}_n^*}(\vec{x}, \vec{y})$ is homogeneous of total degree $4n^2$.

Recall that the scale factor $y_{n+1}^n \left[\prod_{i=1}^n x_i^{2n} y_i^{2n-1}\right]$ was chosen based on the maximum possible degree for the inverse of each variable. The reasoning we used to prove that the maximum possible degree of $\bar{x}_i$ in $Z_{\mathcal{C}_n^*}(\vec{x}, \vec{y})$ must be $2n$ can be used to conclude that the maximum possible degree of x_i must be $2n$ as well. Similarly, the maximum degree of y_i, $1 \leq i \leq n$, and the maximum degree of y_{n+1} are $2n-1$ and n, respectively. In $\tilde{Z}_{\mathcal{C}_n^*}(\vec{x}, \vec{y})$, then, the maximum degree of x_i is $2n + 2n = 4n$, y_i is $(2n-1) + (2n-1) = 4n-2$, and y_{n+1} is $n + n = 2n$. Thus, $\tilde{Z}_{\mathcal{C}_n^*}(\vec{x}, \vec{y})$is a polynomial in x_i^2 of degree $2n$, a polynomial in y_i^2 of degree $2n-1$, and is a polynomial in y_{n+1} of degree $2n$. ★

Lemma 3.6 (Recursive formula for $Z_{\mathcal{C}_n^*}(\vec{x}, \vec{y})$)**.**
If $y_1 = a x_1$, then

$$\frac{Z_{\mathcal{C}_n^*}(\vec{x}, \vec{y})}{Z_{\mathcal{C}_{n-1}^*}(\vec{x}', \vec{y}')} = \sigma\left(a^2\right) \sigma(a\bar{x}_1 y_1) \sigma(a\bar{x}_1 y_{n+1}) \left[\prod_{j=2}^n \sigma(a\bar{x}_1 y_j)^2 \sigma(a\bar{x}_j y_1)^2\right],$$

where $\vec{x}' = \vec{x} \setminus \{x_1\} = \{x_2, x_3, \ldots, x_n\}$ and $\vec{y}' = \vec{y} \setminus \{y_1\} = \{y_2, y_3, \ldots, y_{n+1}\}$.

Proof. Let A be a $\mathcal{C}_n^*$ matrix. Let $\alpha \in \Gamma_n^*$ be the ice state corresponding to A. If the top left corner entry of A were a 0, then the boundary condition on α dictates that the top left corner vertex of α

to be of SW configuration. The weight of this vertex would be $\sigma(ax_1\bar{y}_1) = ax_1(\bar{a}\bar{x}_1) - \bar{a}\bar{x}_1(ax_1) = 1 - 1 = 0$. Then the weight of α would also be 0, and so α would not contribute to $Z_{\mathcal{C}_n^*}(\vec{x}, \vec{y})$.

Thus, we only need to consider the matrices in $\mathcal{C}_n^*$ with a nonzero element in the top left corner. Let A be such a matrix. By definition, the top row of any $\mathcal{C}_n^*$ matrix must be sign-alternating, which forces the first nonzero entry in the row to be 1. In the case when $y_1 = ax_1$, the only matrices in $\mathcal{C}_n^*$ that have nontrivial contribution to $Z_{\mathcal{C}_n^*}(\vec{x}, \vec{y})$ are ones with a 1 in the top left corner, which means that their corresponding ice states must have an EW vertex in the upper left corner. If the first row in A were to have other nonzero entries, then the next nonzero entry must be -1, in order for the row to be sign-alternating. But we know that each column in A is sign-alternating, which means that the first nonzero element in a column must be 1. Therefore, we can conclude that all other entries in the first row of A are 0's. Similarly, there cannot be other nonzero elements in the first column of A, thus it is a column beginning with 1 and followed by 0's. By the half-turn symmetry of the $\mathcal{C}_n^*$ matrices, we know that the last row of A is all 0 except for the last entry, which has value 1 and the last column of A is all 0 except for the last entry. The matrix in Figure 18 shows the form that A must have. The information we know about the "boundary" entries of A, combined with the boundary condition on α, allow us to pinpoint the configurations of vertices that occur on the "outer shell" of α, shown in the ice state in Figure 18.

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & & & & & & 0 \\ 0 & & & & & & 0 \\ \vdots & & & & & & \vdots \\ 0 & & & & & & 0 \\ 0 & & & & & & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix} \quad \Leftrightarrow$$

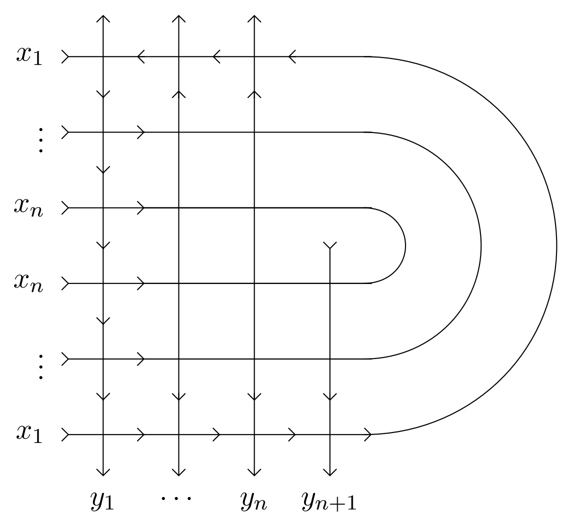


FIGURE 18. "Boundary" of A and "outer shell" of α
With the exception of the vertex in the upper left corner, all vertices in the top row have SE configuration, all vertices in the last row have NW configuration, and all vertices in the first column have NW configuration.

Notice that the "interior" of α is precisely the boundary condition for an ice state corresponding to a $\mathcal{C}^*_{n-1}$ matrix. Then Γ^*_{n-1} gives all possible ways of filling out the "interior" of α. We know that all $\mathcal{C}^*_n$ matrices that contribute nontrivially to $Z_{\mathcal{C}^*_n}(\vec{x},\vec{y})$ have the same "boundary" as A, which means that their corresponding ice states have the same "outer shell" as α. This tells us that $Z_{\mathcal{C}^*_n}(\vec{x},\vec{y})$ is equal to the weight of the "outer shell" of α multiplied by $Z_{\mathcal{C}^*_{n-1}}(\vec{x}',\vec{y}')$. In other words, the ratio $\frac{Z_{\mathcal{C}^*_n}(\vec{x},\vec{y})}{Z_{\mathcal{C}^*_{n-1}}(\vec{x}',\vec{y}')}$ is equal to the weight of the "outer shell".

Since we know the vertex configurations of vertices in the "outer shell", we can compute its weight by breaking the shell into four parts:

- Upper left corner has EW configuration: $\sigma\left(a^2\right)$
- All other vertices in the first row have SE configuration: $\prod_{i=2}^{n}\sigma(a\bar{x}_1 y_i)$
- All vertices in the bottom row have NW configuration: $\prod_{j=1}^{n+1}\sigma(a\bar{x}_1 y_j)$
- The vertices in the left column that have not already been accounted for all have NW configuration: $\prod_{k=2}^{n}\sigma(a\bar{x}_k y_1)^2$.

Putting everything together, we have:

$$\begin{aligned}
&\frac{Z_{\mathcal{C}^*_n}(\vec{x},\vec{y})}{Z_{\mathcal{C}^*_{n-1}}(\vec{x}',\vec{y}')}\\
=&\sigma\left(a^2\right)\left[\prod_{i=2}^{n}\sigma(a\bar{x}_1 y_i)\right]\left[\prod_{j=1}^{n+1}\sigma(a\bar{x}_1 y_j)\right]\left[\prod_{k=2}^{n}\sigma(a\bar{x}_k y_1)^2\right]\\
=&\sigma\left(a^2\right)\sigma(a\bar{x}_1 y_1)\sigma(a\bar{x}_1 y_{n+1})\left[\prod_{i=2}^{n}\sigma(a\bar{x}_1 y_i)^2\right]\left[\prod_{k=2}^{n}\sigma(a\bar{x}_k y_1)^2\right]\\
=&\sigma\left(a^2\right)\sigma(a\bar{x}_1 y_1)\sigma(a\bar{x}_1 y_{n+1})\left[\prod_{j=2}^{n}\sigma(a\bar{x}_1 y_j)^2\sigma(a\bar{x}_j y_1)^2\right] \qquad (3.7)
\end{aligned}$$

★

Lemma 3.8 (Recursive formula for $\tilde{Z}_{\mathcal{C}^*_n}(\vec{x},\vec{y})$)**.**
If $y_1 = ax_1$, *then*

$$\frac{\tilde{Z}_{\mathcal{C}^*_n}(\vec{x},\vec{y})}{\tilde{Z}_{\mathcal{C}^*_{n-1}}(\vec{x}',\vec{y}')} = \sigma\left(a^2\right)\left(ay_1^2-\bar{a}x_1^2\right)\left(ay_{n+1}^2-\bar{a}x_1^2\right)\left[\prod_{j=2}^{n}\left(ay_j^2-\bar{a}x_1^2\right)^2\left(ay_1^2-\bar{a}x_j^2\right)^2\right],$$

where $\vec{x'} = \vec{x}\setminus\{x_1\} = \{x_2,x_3,\dots,x_n\}$ *and* $\vec{y'} = \vec{y}\setminus\{y_1\} = \{y_2,y_3,\dots,y_{n+1}\}$.

Proof. By Equation (3.7), we know the ratio between $Z_{\mathcal{C}_n^*}(\vec{x},\vec{y})$ and $Z_{\mathcal{C}_{n-1}^*}(\vec{x}',\vec{y}')$. Definition 3.4 tells us that

$$\tilde{Z}_{\mathcal{C}_n^*}(\vec{x},\vec{y}) = Z_{\mathcal{C}_n^*}(\vec{x},\vec{y}) \times y_{n+1}^n \left[\prod_{k=1}^{n} x_k^{2n} y_k^{2n-1}\right]$$

and

$$\tilde{Z}_{\mathcal{C}_{n-1}^*}(\vec{x}',\vec{y}') = Z_{\mathcal{C}_{n-1}^*}(\vec{x}',\vec{y}') \times y_{n+1}^{n-1} \left[\prod_{j=2}^{n} x_j^{2n-2} y_j^{2n-3}\right].$$

Combining these two pieces of information allows us to find $\frac{\tilde{Z}_{\mathcal{C}_n^*}(\vec{x},\vec{y})}{\tilde{Z}_{\mathcal{C}_{n-1}^*}(\vec{x}',\vec{y}')}$.

$$\begin{aligned}
&\frac{\tilde{Z}_{\mathcal{C}_n^*}(\vec{x},\vec{y})}{\tilde{Z}_{\mathcal{C}_{n-1}^*}(\vec{x}',\vec{y}')}\\
=&\frac{Z_{\mathcal{C}_n^*}(\vec{x},\vec{y})}{Z_{\mathcal{C}_{n-1}^*}(\vec{x}',\vec{y}')} \times \frac{y_{n+1}^n \left[\prod_{k=1}^{n} x_k^{2n} y_k^{2n-1}\right]}{y_{n+1}^{n-1} \left[\prod_{j=2}^{n} x_j^{2n-2} y_j^{2n-3}\right]}\\
=&\frac{Z_{\mathcal{C}_n^*}(\vec{x},\vec{y})}{Z_{\mathcal{C}_{n-1}^*}(\vec{x}',\vec{y}')} \times x_1^{2n} y_1^{2n-1} y_{n+1} \left[\prod_{k=2}^{n} x_k^2 y_k^2\right]\\
=&\sigma\left(a^2\right)\sigma(a\bar{x}_1 y_1)\sigma(a\bar{x}_1 y_{n+1}) \left[\prod_{j=2}^{n} \sigma\left(a\bar{x}_1 y_j\right)^2 \sigma\left(a\bar{x}_j y_1\right)^2\right] \times x_1^{2n} y_1^{2n-1} y_{n+1} \left[\prod_{k=2}^{n} x_k^2 y_k^2\right]\\
=&\sigma\left(a^2\right) x_1^{2n} y_1^{2n-2} \left[\left(a\bar{x}_1 y_1 - \bar{a}x_1\bar{y}_1\right) y_1\right] \left[\left(a\bar{x}_1 y_{n+1} - \bar{a}x_1\bar{y}_{n+1}\right) y_{n+1}\right] \times\\
&\left[\prod_{j=2}^{n} \left(y_j\left(a\bar{x}_1 y_j - \bar{a}x_1\bar{y}_j\right)\right)^2 \left(x_j\left(a\bar{x}_j y_1 - \bar{a}x_j\bar{y}_1\right)\right)^2\right]\\
=&\sigma\left(a^2\right) x_1^{2n} y_1^{2n-2} \left[a\bar{x}_1 y_1^2 - \bar{a}x_1\right] \left[a\bar{x}_1 y_{n+1}^2 - \bar{a}x_1\right] \left[\prod_{j=2}^{n} \left(a\bar{x}_1 y_j^2 - \bar{a}x_1\right)^2 \left(a y_1 - \bar{a}x_j^2\bar{y}_1\right)^2\right]\\
=&\sigma\left(a^2\right) \left[x_1(a\bar{x}_1 y_1^2 - \bar{a}x_1)\right] \left[x_1(a\bar{x}_1 y_{n+1}^2 - \bar{a}x_1)\right] \left[x_1^{2n-2} y_1^{2n-2} \prod_{j=2}^{n} \left(a\bar{x}_1 y_j^2 - \bar{a}x_1\right)^2 \left(a y_1 - \bar{a}x_j^2\bar{y}_1\right)^2\right]\\
=&\sigma\left(a^2\right) \left[a y_1^2 - \bar{a}x_1^2\right] \left[a y_{n+1}^2 - \bar{a}x_1^2\right] \left[\prod_{j=2}^{n} \left(x_1\left(a\bar{x}_1 y_j^2 - \bar{a}x_1\right)\right)^2 \left(y_1\left(a y_1 - \bar{a}x_j^2\bar{y}_1\right)\right)^2\right]\\
=&\sigma\left(a^2\right) \left[a y_1^2 - \bar{a}x_1^2\right] \left[a y_{n+1}^2 - \bar{a}x_1^2\right] \left[\prod_{j=2}^{n} \left(a y_j^2 - \bar{a}x_1^2\right)^2 \left(a y_1^2 - \bar{a}x_j^2\right)^2\right]
\end{aligned}$$

★

Definition 3.9 (Symmetric function)**.**
Let f be a function in the variables $z_1, z_2, \ldots, z_n$. Then f is called a symmetric function *in the variables $z_1, z_2, \ldots, z_n$ if it satisfies:*

$$f(z_{s(1)}, z_{s(2)}, \ldots, z_{s(n)}) = f(z_1, z_2, \ldots, z_n)$$

for all $s \in S_n$.

In other words, a function f is symmetric in the variables $z_1, z_2, \ldots, z_n$ if permuting the variables in the function gives back f. See Example 3.10 for an illustration of this concept.

Example 3.10 (Symmetric Functions)**.**
Consider the function $f(x, y) = x^2 + y^2$. Since f only has two variables, there is only one way to permute its elements, from (x, y) to (y, x). Then $f(x, y) = x^2 + y^2 = y^2 + x^2 = f(y, x)$. Therefore f is symmetric in the variables x, y.

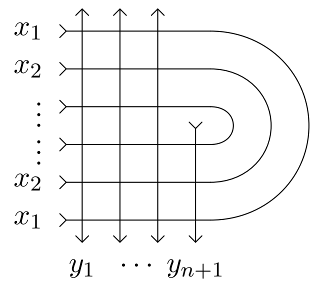
Lemma 3.11.
The modified partition function $\tilde{Z}_{\mathcal{C}_n^*}(\vec{x}, \vec{y})$ is symmetric in the variables $x_1, \ldots, x_n$.

Proof. Recall that $\tilde{Z}_{\mathcal{C}_n^*}(\vec{x}, \vec{y})$ is equal to $Z_{\mathcal{C}_n^*}(\vec{x}, \vec{y})$ scaled by $y_{n+1}^n \left[\prod_{i=1}^n x_i^{2n} y_i^{2n-1}\right]$. In the scale factor, all x_i's have the same power, so it's clear that it's symmetric with respect to $x_1, x_2, \ldots, x_n$. The product of symmetric functions is symmetric, so if we can show that $Z_{\mathcal{C}_n^*}(\vec{x}, \vec{y})$ is symmetric with respect to the x_i's, then it follows that the modified partition function is also.

To show that $Z_{\mathcal{C}_n^*}(\vec{x}, \vec{y})$ is invariant under all permutations of the variables $x_1, x_2, \ldots, x_n$, it is sufficient to show that $Z_{\mathcal{C}_n^*}(\vec{x}, \vec{y})$ is invariant under the swapping of any x_i and x_j for $i \neq j$, since all permutations can be built from such swaps (every permutation can be written as a product of transpositions). To show that all swappings are possible, it is sufficient to show that the swapping of any two adjacent variables is possible. By "adjacent" we mean that the indices i and j have a difference of 1. This is the case because any swapping can be achieved by successively swapping adjacent variables. Without loss of generality, we will show that $Z_{\mathcal{C}_n^*}(\vec{x}, \vec{y})$ is invariant under the swapping of x_1 and x_2 using the Yang-Baxter Equation.

In Definition 1.20, we defined the partition function of a class of matrices to be the sum over all weights of the corresponding ice states. For $\mathcal{C}_n^*$ matrices, this means summing over the weights of all elements in Γ_n^*. Since Γ_n^* consists of all possible ways to fill in the $\mathcal{C}_n^*$ boundary condition, a natural way to depict the partition function $Z_{\mathcal{C}_n^*}(\vec{x},\vec{y})$ is:

$$Z_{\mathcal{C}_n^*}(\vec{x},\vec{y}) = \sum_{\substack{\text{all possible}\\ \text{fillings}}}$$

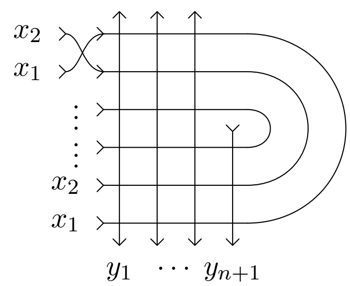


,

where the picture of the ice state represents its weight.

Consider the following function:

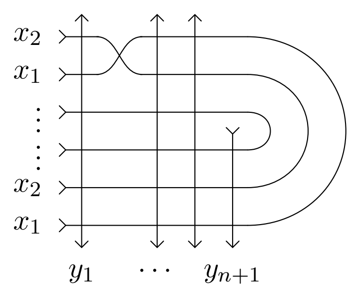
$$\sum_{\substack{\text{all possible}\\ \text{fillings}}} \tag{3.12}$$



.

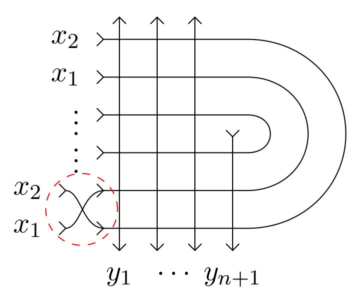
The Yang-Baxter Equation tells us that Equation (3.12) is equal to

$$\sum_{\substack{\text{all possible}\\ \text{fillings}}} \tag{3.13}$$



since all we did was "pass through" the twist. We can continue this process of "passing through" the twist, and each time the resulting function would be equal to the one before by the Yang-Baxter Equation. Eventually, we would arrive at the following function:

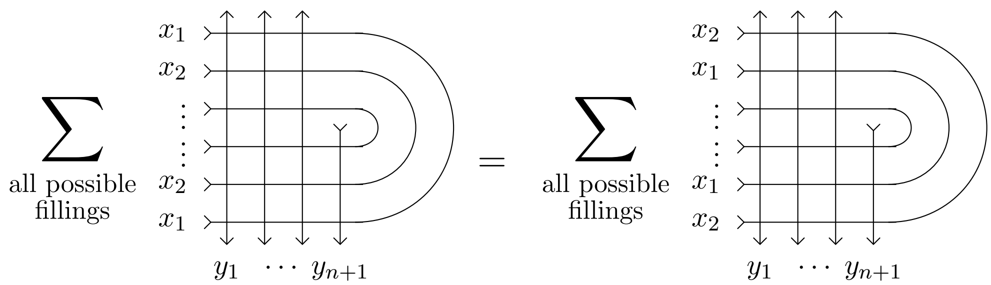
$$\sum_{\substack{\text{all possible}\\ \text{fillings}}} \tag{3.14}$$



.

Note that the circled vertex in Equation (3.14) must have the configuration shown. The boundary condition dictates that the two edges on the left both point inward, which forces the two edges on the right to point outward.

In both Equations (3.12) and (3.14), since every possible filling has the same twist configuration, its weight can be factored out from the function. Notice that in Equations (3.12) and (3.14), the twist has the same configuration, and so the weight that is factored out is the same for both functions and can be canceled out. What we have then is the following equality, which follows from the fact that Equations (3.12) and (3.14) are equal:



(3.15)

The function on the left of Equation (3.15) is equal to $Z_{\mathcal{C}_n^*}(\vec{x},\vec{y}) = Z_{\mathcal{C}_n^*}((x_1, x_2, \ldots, x_n), \vec{y})$ while the function on the right is equal to $Z_{\mathcal{C}_n^*}((x_2, x_1, \ldots, x_n), \vec{y})$. This tells us that $Z_{\mathcal{C}_n^*}(\vec{x},\vec{y})$ is invariant under the swapping of x_1 and x_2. ★

Lemma 3.16.

The modified partition function $\tilde{Z}_{\mathcal{C}_n^*}(\vec{x},\vec{y})$ is symmetric in the variables $y_1, \ldots, y_n$.

Proof. The proof for this is analogous to the proof of Lemma 3.11. Instead of sending a twist through two adjacent horizontal lines, do the same for two adjacent vertical lines. ★

Lemma 3.17.

The partition function $Z_{\mathcal{C}_n^*}(\vec{x},\vec{y})$ is invariant under the simultaneous replacements $x_i \mapsto \bar{x}_i$ and $y_j \mapsto \bar{y}_j$, for $i \in \{1, 2, \ldots, n\}$ and $j \in \{1, 2, \ldots, n+1\}$. In other words, $Z_{\mathcal{C}_n^*}(\vec{x},\vec{y}) = Z_{\mathcal{C}_n^*}(\vec{\bar{x}},\vec{\bar{y}})$ where $\vec{\bar{x}} = (\bar{x}, \bar{x}_2, \ldots, \bar{x}_n)$ and $\vec{\bar{y}} = (\bar{y}, \bar{y}_2, \ldots, \bar{y}_{n+1})$. For the modified partition function one has

$$\tilde{Z}_{\mathcal{C}_n^*}(\vec{\bar{x}},\vec{\bar{y}}) = \bar{y}_{n+1}^{2n} \left[\prod_{i=1}^{n} \bar{x}_i^{4n} \bar{y}_i^{4n-2} \right] \tilde{Z}_{\mathcal{C}_n^*}(\vec{x},\vec{y}).$$

Proof. Let A be a $\mathcal{C}_n^*$ matrix. Let $\alpha \in \Gamma_n^*$ be the ice state corresponding to A. Define $f : \Gamma_n^* \to \Gamma_n^*$ as the map that first reflects an ice state through the horizontal bisecting line and then rotates the half-line with the spectral parameter y_{n+1} by 180°. Figure 19 illustrates the function f for a Γ_2^* ice state.

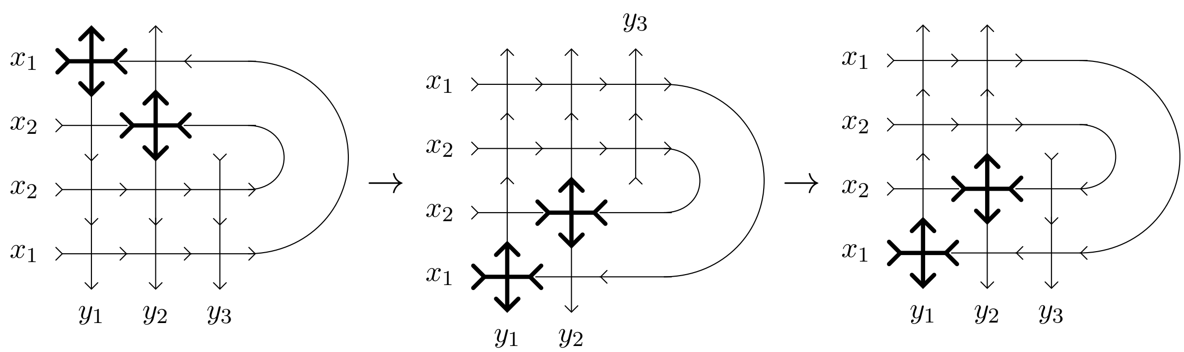


FIGURE 19. Illustration of f
First we reflect the matrix, then we translate the upper half column to a lower half column.

Notice that f is its own inverse and is therefore a bijection. Then the image of f is Γ_n^*. From this, we know that the sum of the weights of $\alpha \in \Gamma_n^*$ is equal to the sum of the weights of $f(\alpha)$. By definition, the sum of the weights of $\alpha \in \Gamma_n^*$ is equal to $Z_{\mathcal{C}_n^*}(\vec{x}, \vec{y})$. Our claim is that the sum of the weights of $f(\alpha)$ is equal to $\mathcal{C}^*(n; \vec{\bar{x}}, \vec{\bar{y}})$. We will prove this by showing that for any vertex configuration v, f maps v to a configuration whose weight is obtained from the weight of v using the replacements $x_i \mapsto \bar{x}_i$, $y_i \mapsto \bar{y}_i$; refer to Table 1.

By definition, we have

$$\tilde{Z}_{\mathcal{C}_n^*}(\vec{x}, \vec{y}) = y_{n+1}^n \left[\prod_{i=1}^{n} x_i^{2n} y_i^{2n-1} \right] Z_{\mathcal{C}_n^*}(\vec{x}, \vec{y}). \tag{3.18}$$

It follows that

$$\tilde{Z}_{\mathcal{C}_n^*}(\vec{\bar{x}}, \vec{\bar{y}}) = \bar{y}_{n+1}^n \left[\prod_{i=1}^{n} \bar{x}_i^{2n} \bar{y}_i^{2n-1} \right] Z_{\mathcal{C}_n^*}(\vec{\bar{x}}, \vec{\bar{y}}). \tag{3.19}$$

Our earlier result tells us that $Z_{\mathcal{C}_n^*}(\vec{x}, \vec{y}) = Z_{\mathcal{C}_n^*}(\vec{\bar{x}}, \vec{\bar{y}})$. This fact, combined with Equations (3.18) and (3.19), allows us to conclude that

$$\tilde{Z}_{\mathcal{C}_n^*}(\vec{\bar{x}}, \vec{\bar{y}}) = \bar{y}_{n+1}^{2n} \left[\prod_{i=1}^{n} \bar{x}_i^{4n} \bar{y}_i^{4n-2} \right] \tilde{Z}_{\mathcal{C}_n^*}(\vec{x}, \vec{y}).$$ ★

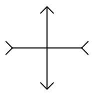
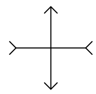
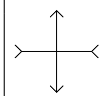
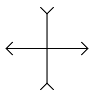
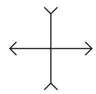
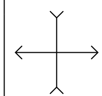
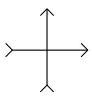
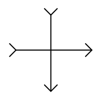
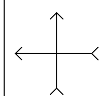
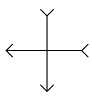
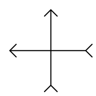
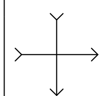
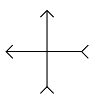
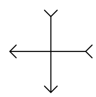
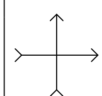
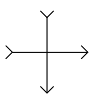
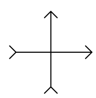
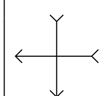
Original	After horizontal reflection	After horizontal reflection and 180° rotation (vertices in half column only)
 $\sigma\left(a^2\right)$	 $\sigma\left(a^2\right)$	 $\sigma\left(a^2\right)$
 $\sigma\left(a^2\right)$	 $\sigma\left(a^2\right)$	 $\sigma\left(a^2\right)$
 $\sigma(ax_i\bar{y}_j)$	 $\sigma(a\bar{x}_iy_j)$	 $\sigma(a\bar{x}_iy_j)$
 $\sigma(ax_i\bar{y}_j)$	 $\sigma(a\bar{x}_iy_j)$	 $\sigma(a\bar{x}_iy_j)$
 $\sigma(a\bar{x}_iy_j)$	 $\sigma(ax_i\bar{y}_j)$	 $\sigma(ax_i\bar{y}_j)$
 $\sigma(a\bar{x}_iy_j)$	 $\sigma(ax_i\bar{y}_j)$	 $\sigma(ax_i\bar{y}_j)$

TABLE 1. Effects of f on a vertex with spectral parameter $x_i\bar{y}_j$

Lemma 3.20.

The modified partition function $\tilde{Z}_{\mathcal{C}_n^*}(\vec{x},\vec{y})$ is invariant under the replacement $a \mapsto -a$.

Proof. Because the parameter a is not present in the scale factor of $Z_{\mathcal{C}_n^*}(\vec{x},\vec{y})$ that results in $\tilde{Z}_{\mathcal{C}_n^*}(\vec{x},\vec{y})$, if we can prove that $Z_{\mathcal{C}_n^*}(\vec{x},\vec{y})$ is invariant under the replacement $a \mapsto -a$, then it follows that its modified version is also. We will show that for each matrix $A \in \mathcal{C}_n^*$, the weight of the corresponding ice state $\alpha \in \Gamma_n^*$ is a polynomial in a^2. From the proof of Lemma 3.5, we know that the weight of α has the following form:

$$wt(\alpha) = \left(a^2 - \bar{a}^2\right)^{b_\alpha} (a\bar{x}_iy_j - \bar{a}\bar{x}_iy_j)^{c_\alpha} (a\bar{x}_iy_j - \bar{a}x_i\bar{y}_j)^{d_\alpha}.$$

The factors of $\left(a^2 - \bar{a}^2\right)$ are contributed by vertices of configuration EW or NS. Since this factor is itself a polynomial in a^2, $(a^2 - \bar{a}^2)^{b_\alpha}$ will be a polynomial in a^2 independent of the value of b_α (the number of EW and NS vertices in α). The factors $(a\bar{x}_iy_j - \bar{a}\bar{x}_iy_j)$ and $(a\bar{x}_iy_j - \bar{a}x_i\bar{y}_j)$ come from the remaining four vertex configurations, all of which correspond to an entry of 0 in A. If we can show that $c_\alpha + d_\alpha$ is even, then it follows that the weight of α is a polynomial in a^2. This is equivalent to proving that there are an even number of 0's in the half of A that is encoded by α. For ease of notation, we will refer to this pseudo-matrix as A'.

We know that A' has n columns with $2n$ entries and a half column with n entries. By definition of $\mathcal{C}_n^*$ matrix, each column in A sums to 1. This means that in each column, there must be at least one 1 and that there is one more 1 than -1. Therefore, we can conclude that p_i, the number of nonzero vertices in a full column i, is odd for all i. Thus, any (full) column i in A' has $2n - p_i$ zeros. We know that the half-column must sum to 0, which means there are an equal number of 1's and -1's in the matrix. From this we know that q, the number of nonzero vertices in the half column, is even. The half column contains $n - q$ zero entries. Putting this information together, there are $(\sum_{i=1}^{n} 2n - p_i) + (n - q)$ zero entries in A'. Regardless of the parity of n, we know that $2n$ is even. And since p_i is odd, the difference $2n - p_i$ is odd for all i. We now break the proof up into two cases based on the parity of n:

(1) n is even

Then $(\sum_{i=1}^{n} 2n - p_i)$ is the sum of an even number of odd numbers, so it is even. Since $n - q$ is the difference of two even numbers, it is even as well. The sum of two even numbers is even, so there are an even number of zeros in A'.

(2) n is odd

The expression $(\sum_{i=1}^{n} 2n - p_i)$ is an sum of an odd number of odd numbers, so it is odd. Since $n - q$ is the difference of an even number and an odd number, it is odd as well. The sum of two odd numbers is even, so there are an even number of zeros in A'. ★

Theorem 3.21.

The modified partition function $\tilde{Z}_{\mathcal{C}_n^*}(\vec{x}, \vec{y})$ has the following properties:

(a) $\tilde{Z}_{\mathcal{C}_n^*}(\vec{x}, \vec{y})$ is symmetric in the variables $x_1, \ldots, x_n$.

(b) $\tilde{Z}_{\mathcal{C}_n^*}(\vec{x}, \vec{y})$ is symmetric in the variables $y_1, \ldots, y_n$.

(c) $Z_{\mathcal{C}_n^*}(\vec{x}, \vec{y})$ is invariant under the simultaneous replacements $x_i \mapsto \bar{x}_i$ and $y_i \mapsto \bar{y}_i$, for $i \in \{1, 2, \ldots, n\}$ In other words, $Z_{\mathcal{C}_n^*}(\vec{x}, \vec{y}) = Z_{\mathcal{C}_n^*}(\vec{\bar{x}}, \vec{\bar{y}})$ where $\vec{\bar{x}} = (\bar{x}, \bar{x}_2, \ldots, \bar{x}_n)$ and $\vec{\bar{y}} = (\bar{y}, \bar{y}_2, \ldots, \bar{y}_n)$. For the modified partition function one has

$$\tilde{Z}_{\mathcal{C}_n^*}(\vec{\bar{x}}, \vec{\bar{y}}) = \bar{y}_{n+1}^{2n} \left[\prod_{i=1}^{n} \bar{x}_i^{4n} \bar{y}_i^{4n-2} \right] \tilde{Z}_{\mathcal{C}_n^*}(\vec{x}, \vec{y}).$$

(d) $\tilde{Z}_{\mathcal{C}_n^*}(\vec{x},\vec{y})$ *is invariant under the replacement* $a \mapsto -a$.

(e) $\tilde{Z}_{\mathcal{C}_n^*}(\vec{x},\vec{y})$ *is a homogeneous polynomial in the variables* x_i *and* y_i *of total degree* $4n^2$. *For each fixed* $i \in \{1,\ldots,n\}$, *the modified partition function is a polynomial in* x_i^2 *of degree* $2n$ *and a polynomial in* y_i^2 *of degree* $2n-1$. *It is a polynomial in* y_{n+1}^2 *of degree* n.

(f) If $y_1 = ax_1$, *then*

$$\frac{Z_{\mathcal{C}_n^*}(\vec{x},\vec{y})}{Z_{\mathcal{C}_{n-1}^*}(\vec{x}',\vec{y}')} = \sigma\left(a^2\right)\sigma(a\bar{x}_1y_1)\sigma(a\bar{x}_1y_{n+1})\left[\prod_{j=2}^{n}\sigma(a\bar{x}_1y_j)^2\sigma(a\bar{x}_jy_1)^2\right],$$

$$\frac{\tilde{Z}_{\mathcal{C}_n^*}(\vec{x},\vec{y})}{\tilde{Z}_{\mathcal{C}_{n-1}^*}(\vec{x}',\vec{y}')} = \sigma\left(a^2\right)\left(ay_1^2-\bar{a}x_1^2\right)\left(ay_{n+1}^2-\bar{a}x_1^2\right)\left[\prod_{j=2}^{n}\left(ay_j^2-\bar{a}x_1^2\right)^2\left(ay_1^2-\bar{a}x_j^2\right)^2\right]$$

where $\vec{x}' = \vec{x}\setminus\{x_1\} = \{x_2,x_3,\ldots,x_n\}$ *and* $\vec{y}' = \vec{y}\setminus\{y_1\} = \{y_2,y_3,\ldots,y_{n+1}\}$.

Proof. Refer to Lemmas 3.11, 3.16, 3.17, 3.20, 3.5, 3.6, and 3.8. ★

At the beginning of this section, we alluded to a lemma used by Razumov and Stroganov that motivated much of the work in this section. Below we give the equivalent result for $\mathcal{C}_n^*$.

Theorem 3.22 (Equivalence of partition functions).
If two functions $\tilde{Z}_{\mathcal{C}_n^*}(\vec{x},\vec{y})$ *and* $\tilde{Z}'_{\mathcal{C}_n^*}(\vec{x},\vec{y})$ *have the properties described in Lemmas 3.11, 3.16, 3.5, and 3.8, have the same recursive relation when* $y_{n+1} = \bar{a}x_n$, *and their corresponding lead polynomials* $S(n;x_n,y_{n+1})$ *and* $S'(n;x_n,y_{n+1})$ *coincide, then* $\tilde{Z}_{\mathcal{C}_n^*}(\vec{x},\vec{y}) = \tilde{Z}'_{\mathcal{C}_n^*}(\vec{x},\vec{y})$.

4. Subclasses of $\mathcal{C}_n^*$ Matrices

Our goal is to relate $\tilde{Z}_{\mathcal{C}_n^*}(\vec{x},\vec{y})$ to partition functions of ASMs whose enumerations are known. If we can find a representation of $\tilde{Z}_{\mathcal{C}_n^*}(\vec{x},\vec{y})$ as a combination of partition functions of other classes of ASMs, and if we can relate $Z_{\mathcal{C}_n^*}(\vec{x},\vec{y})$ to the x-enumeration problem for $\mathcal{C}_n^*$ (much like Kuperberg did in Theorem 1.21), then we will be able to enumerate the $\mathcal{C}_n^*$ matrices.

We have already verified the appropriate symmetry conditions for $\tilde{Z}_{\mathcal{C}_n^*}(\vec{x},\vec{y})$, so our next step is to find an explicit formula for the lead coefficient of the polynomial. The lead coefficient $S(n;x_n,y_{n+1})$ in Theorem 3.22 is the coefficient on the term with the highest x_i term for $i = 1,2,\ldots,n-1$. For an individual $\mathcal{C}_n^*$ matrix, maximizing the power of x_i is equivalent to minimizing the number of nonzero elements in rows i and $2n+1-i$ (these are the rows are correspond to the x_i spectral parameter). For ASMs, permutation matrices were the only matrices that contributed the term in which all x_i were of the highest degree. In our quest to find such a formula, we found it useful to define three subclasses of $\mathcal{C}_n^*$ matrices, each of which minimizes the the number of nonzero elements in specific rows. It is not surprising that all three definitions make use of the symmetric group, S_n, from which permutation matrices are defined.

4.1. **Class 1 Matrices.** Let $F_n = S_n \times (\mathbb{Z}/2\mathbb{Z})^n$. Define f to be a map from F_n to $\mathcal{C}_n^*$. If $(s, z_1, z_2, \ldots, z_n)$ is an element of F_n, then $f((s, z_1, z_2, \ldots, z_n)) = A_{(2n)\times(2n+1)}$ is constructed using the following algorithm:

(1) Place all 0's in the middle column (column $n+1$).
(2) Take the permutation matrix associated with s and place it in the upper left corner of A. Notice that because $s \in S_n$, this permutation matrix has dimension $n \times n$.
(3) For each row i in the permutation matrix, the value of z_i tells us whether this row is "flipped". If $z_i = 0$, nothing is done to row i and the first n entries of row $2n+1-i$ are filled with 0's. If $z_i = 1$, then the row is flipped, which means that the contents of the ith row of the permutation matrix is "flipped" across the horizontal middle of A and placed in row $2n+1-i$ instead and the first n entries in row i are filled with 0's.
(4) The half-turn symmetry of the $\mathcal{C}_n^*$ matrix allows us to fill in the rest of the matrix.

Notice that in filling out the first n columns of the matrix, only half of the rows have a 1 while the others have only 0's. Considering the way we have filled out the matrix thus far, the first n entries in row i is all zero if and only if the first n entries in row $2n+1-i$ has a 1. Then, when we take half-turn symmetry into account, the last n entries in row i will have a 1 precisely when the first n entries row $2n+1-i$ has a 1, which is only when the first n entries row i is all zero. For all matrices in the range of map f, every row and column (except $n+1$) has exactly one 1. See Example 4.1 for an illustration of the map f.

Example 4.1.

We will show how the function f acts on $((132), 0, 1, 1) \in F_3 = S_3 \times (\mathbb{Z}/2\mathbb{Z})^3$. Note that the resulting matrix A will have dimension 6×7. The permutation (132) corresponds to the permutation matrix

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

(1) Begin by placing 0*'s in the middle column (column 4) and placing the permutation matrix at the top left corner of* A*.*

$$\begin{bmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ & \cdots & & 0 & \cdots \\ & \cdots & & 0 & \cdots \\ & \cdots & & 0 & \cdots \end{bmatrix}$$

(2) Since $z_1 = 0$*, we leave row 1 alone fill the first half of row 6 with* 0*'s.*

$$\begin{bmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ & \cdots & & 0 & \cdots \\ & \cdots & & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \end{bmatrix}$$

(3) Since $z_2 = 1$*, we "flip" row 2 of the permutation matrix down to row 5 and fill the first half of row 2 with* 0*'s.*

$$\begin{bmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ & \cdots & & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \end{bmatrix}$$

(4) *Since* $z_3 = 1$*, we "flip" row 3 of the permutation matrix down to row 4 and fill the first half of row 3 with* 0*'s.*

$$\begin{bmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \end{bmatrix}$$

(5) *Fill in the rest of matrix* A *using the half-turn symmetry condition* $a_{i,j} = a_{2n+1-i,2n+2-j}$.

$$\begin{bmatrix} 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} \\ \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 \end{bmatrix}$$

4.2. **Class 2 Matrices.** For $n \in \mathbb{N}$, let $[n] = \{1, 2, \ldots, n\}$. Let $G_n = [n] \times [n] \times [n-1] \times S_{n-2} \times (\mathbb{Z}/2\mathbb{Z})^{n-2}$. Define H_n to be the subset of G_n such that $x \neq y$. Define g to be a map from H_n to $\mathcal{C}_n^*$. If $(x, y, z, s, z_1, z_2, \ldots, z_{n-2})$ is an element of H_n, then $g((x, y, z, s, z_1, z_2, \ldots, z_{n-2})) = B_{(2n)\times(2n+1)}$ is constructed using the following algorithm:

(1) Place -1's in the nth and $n+1$st entries of the middle column.
(2) In the middle column, place a 1 in row z. The half-turn symmetry of the matrix tells us that there must be a 1 in row $2n+1-z$ as well.
(3) The first n entries of row n contains a single 1, in the xth column, and all other entries are 0.
(4) The first n entries of row $n+1$ contains a single 1, in the yth column, and all other entries are 0.
(5) Crossing out all rows and columns containing a nonzero value, we are left with an empty $(n-2) \times (n-2)$ matrix in the upper left corner of B. Fill the left half of B using $s, z_1, z_2, \ldots, z_{n-2}$ as we did for Class 1 matrices.
(6) Complete the matrix using half-turn symmetry.

See Example 4.2 for an illustration of g. Notice that with the exception of column $n+1$, row n, and row $n+1$, each row and column of B has exactly one 1.

Example 4.2.

We will show how the function h acts on $(1, 2, 1, (1), 1) \in H_3 \subset G_3 = [3] \times [3] \times [2] \times S_1 \times (\mathbb{Z}/2\mathbb{Z})$. The element $(1) \in S_1$ corresponds to the permutation matrix $\begin{bmatrix} 1 \end{bmatrix}$.

(1) *We begin by filling out column 4. Rows 3 and 4 must have entries -1. Since $z = 1$, row 1 and row 6 have entries 1. All other entries are 0.*

$$\begin{bmatrix} \cdots & 1 & \cdots \\ \cdots & 0 & \cdots \\ \cdots & -1 & \cdots \\ \cdots & -1 & \cdots \\ \cdots & 0 & \cdots \\ \cdots & 1 & \cdots \end{bmatrix}$$

(2) *Since $x = 1$ and $y = 2$, there is a 1 in the first entry of row 3 and a 1 in the second entry of row 4. All other entries in the left half of rows n and $n+1$ are 0's.*

$$\begin{bmatrix} & \cdots & & 1 & \cdots \\ & \cdots & & 0 & \cdots \\ 1 & 0 & 0 & -1 & \cdots \\ 0 & 1 & 0 & -1 & \cdots \\ & \cdots & & 0 & \cdots \\ & \cdots & & 1 & \cdots \end{bmatrix}$$

(3) *Crossing out the columns and rows that already contain nonzero elements, we are left with a single unfilled entry in the upper left corner of B, entry $b_{2,3}$. We place the permutation matrix $\begin{bmatrix} 1 \end{bmatrix}$ in this entry. Since $z_1 = 1$, we "flip" this 1 to entry $b_{5,3}$ and place a 0 in entry $b_{2,3}$. All unfilled entries in the left half of B are filled with 0.*

$$\begin{bmatrix} 0 & 0 & 0 & 1 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & -1 & \cdots \\ 0 & 1 & 0 & -1 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \end{bmatrix}$$

(4) Now we can complete matrix B using half-turn symmetry.

$$\begin{bmatrix} 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 \\ \mathbf{1} & 0 & 0 & \mathbf{-1} & 0 & \mathbf{1} & 0 \\ 0 & \mathbf{1} & 0 & \mathbf{-1} & 0 & 0 & \mathbf{1} \\ 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 \end{bmatrix}$$

4.3. **Class 2A and Class 2B Matrices.** We will now partition Class 2 matrices into two types, Class 2A and Class 2B, based on the position of the 1 in the middle column. If the 1 is not in the first row, then the matrix is Class 2A. If the 1 is in the first row, then the matrix is Class 2B. Matrix B from Example 4.2 is a Class 2B matrix. See Figure 20 for an example of a Class 2A matrix.

$$\begin{bmatrix} 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} & \mathbf{-1} & 0 & 0 & \mathbf{1} \\ \mathbf{1} & 0 & 0 & \mathbf{-1} & \mathbf{1} & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 \end{bmatrix}$$

FIGURE 20. A Class 2A matrix of rank 3

5. $M(n)$ Coefficient

Referencing Lemma 3.5, we see that the maximum degree on x_i in $\tilde{Z}_{\mathcal{C}_n^*}(\vec{x},\vec{y})$ is $4n$ for $i = 1, 2, \ldots, n$. Singling out the term in $\tilde{Z}_{\mathcal{C}_n^*}(\vec{x},\vec{y})$ with maximum degree in all x_i's, we give the coefficient on this term a name.

Definition 5.1 ($M(n)$ Coefficient)**.**
Singling out from $\tilde{Z}_{\mathcal{C}_n^*}(\vec{x},\vec{y})$ the term of maximal degree in the variables x_i, we write

$$\tilde{Z}_{\mathcal{C}_n^*}(\vec{x},\vec{y}) = \left[\prod_{i=1}^{n} x_i^{4n}\right] M(n) + \ldots .$$

A entry of 1 or -1 contributes $\sigma\left(a^2\right) = a^2 - \bar{a}^2$ to the matrix's weight (refer to Figure 10) and no x_i term. If we we want find the maximum degree term in the variables x_i, we must only consider those $\mathcal{C}_n^*$ matrices with minimum number of 1 and -1 entries. The definition of $\mathcal{C}_n^*$ matrices requires that each row sums to 1, and so each row must have a minimum of one 1. With the exception of the middle column, each column must also sum to 1, and so they must also have a minimum of one 1. The middle column must sum to 0, hence we want the column to be all 0s. The $\mathcal{C}_n^*$ matrices that contribute to $M(n)$ are those with one 1 in every row and column, with the exception that the middle column is all 0s. Such matrices are precisely the Class 1 matrices that are defined above. Thus, we only need to consider Class 1 matrices when calculating $M(n)$.

Consider the four vertex configurations that correspond to a 0 entry in the matrix: SW, NE, SE, and NW. The weight of each configuration is a polynomial with two terms. Assume that the vertex is in a row with spectral parameter x_i and a column with spectral parameter y_j. Both SW and NE configurations contribute $\sigma(ax_i\bar{y}_j) = (ax_i\bar{y}_j - \bar{a}\bar{x}_iy_j)$ while SE and NW both contribute $\sigma(a\overline{x_i\overline{y}_j}) = \sigma(a\bar{x}_iy_j) = (a\bar{x}_iy_j - \bar{a}x_i\bar{y}_j)$. Since we are concerned with the coefficient of the term with maximal power in variables x_i, we only care about the term containing x_i. We can ignore the $\bar{y}_j$ because maximizing the the degree of the x_i variables forces the term to have maximum degree in the $\bar{y}_j$ variables. Recall that the scaled factor of the modified partition function is defined to offset the maximum possible degrees in all $\bar{y}_j$ terms. Thus, the term with maximum degree in variables x_i in $\tilde{Z}_{\mathcal{C}_n^*}(\vec{x},\vec{y})$ contains no y_j terms since they canceled with the scaled factor. Thus, we only care about the a part of a vertex's contribution. When calculating an ice model's contribution to $M(n)$, SW and NE contribute a to the weight of the model while SE and NW contribute $-\bar{a}$.

Lemma 5.2.

For any $n \in \mathbb{N}$,

$$\sigma(a^{4n}) = \sigma\left(a^2\right)(a^{4n-2} + a^{4n-6} + a^{4n-10} + \cdots + a^2 + \bar{a}^2 + \cdots + \bar{a}^{4n-10} + \bar{a}^{4n-6} + \bar{a}^{4n-2}).$$

Proof. We will begin with the right hand side of the equation and show that it is equal to the left hand side. By definition of σ, we know that $\sigma(a^2) = a^2 - \bar{a}^2$, so we can substitute $a^2 - \bar{a}^2$ for $\sigma(a^2)$ in the expression to get:

$$(a^2 - \bar{a}^2)(a^{4n-2} + a^{4n-6} + a^{4n-10} + \cdots + a^2 + \bar{a}^2 + \cdots + \bar{a}^{4n-10} + \bar{a}^{4n-6} + \bar{a}^{4n-2}).$$

Multiplying the two polynomials and writing out the terms, we have:

$$\begin{aligned}&(a^{4n} + a^{4n-4} + a^{4n-8} + \cdots + a^4 + 1 + \bar{a}^4 + \cdots + \bar{a}^{4n-12} + \bar{a}^{4n-8} + \bar{a}^{4n-4})\\ &-(a^{4n-4} + a^{4n-8} + a^{4n-12} + \cdots + 1 + \bar{a}^4 + \cdots + \bar{a}^{4n-8} + \bar{a}^{4n-4} + \bar{a}^{4n}).\end{aligned}$$

The terms can be rearranged to this form:

$$a^{4n} + (a^{4n-4} + a^{4n-8} + \cdots + \bar{a}^{4n-8} + \bar{a}^{4n-4}) - (a^{4n-4} + a^{4n-8} + \cdots + \bar{a}^{4n-8} + \bar{a}^{4n-4}) - \bar{a}^{4n}.$$

It is clear that the above expression simplies to $a^{4n} - \bar{a}^{4n}$, which is equal to $\sigma(a^{4n})$. ★

For our next lemma, it is useful to group together the four vertex configurations that correspond to a 0 in a matrix and to pinpoint some properties about these four vertex configurations.

Definition 5.3 (Zero Vertices)**.**

Call the set of vertex configurations that correspond to an entry 0 *in a matrix* zero vertices*. Consulting Figure 10, we see that this set consist of SW, NE, SE, and NW configurations.*

One feature that all zero vertices have in common is that along both axises, the direction of the arrow does not change. Consulting Figure 10, we see that for a zero vertex, the left edge points left if and only if the right edge points left. Similarly, the top edges points down if and only if the bottom edge points down. This means that if we know a vertex corresponds to an entry of 0, then we can determine the vertex configuration by knowing only one of the edges from each axis. Knowing this, we can break the zero vertices into groups based the direction of the edges along one of the axises.

Definition 5.4.

Define a vertex configuration to be northernly *if its top edge points toward the vertex and its bottom edge points away from the vertex. Define a vertex configuration to be* southernly *if its top edge points away from the vertex and its bottom edge points toward the vertex. Define a vertex configuration to be* westernly *if its left edge points toward the vertex and its right edge points away from the vertex. Define a vertex configuration to be* easternly *if its left edge points away from the vertex and its right edge points toward the vertex.*

Note that the above definitions only apply to zero vertices. Both EW and NS vertex configuration have the feature that along any axis, both edge point toward or both edges point away from the vertex.

Lemma 5.5.

The coefficient $M(n)$ is given by the formula

$$M(n) = \prod_{i=1}^{n} \sigma(a^{4i}).$$

Proof. We will prove this lemma using recursion. By calculating the modified partition function $\tilde{Z}_{\mathcal{C}_1^*}(\vec{x}, \vec{y})$, we know that $M(1) = \sigma(a^4)$. We will show that for $n > 1$, we have

$$M(n) = \sigma(a^{4n})M(n-1).$$

It follows from the above two facts that $M(n) = \prod_{i=1}^{n} \sigma(a^{4i})$.

Our first objective is to find a way to relate Class 1 $\mathcal{C}_n^*$ matrices to Class 1 $\mathcal{C}_{n-1}^*$ matrices. Specifically, we will partition the set of Class 1 $\mathcal{C}_n^*$ matrices into groups based on the position of the 1 in the first row. Given the column number of the 1 in the first row of a $\mathcal{C}_n^*$ matrix, we delete the first and last row, and the column containing the 1. What is left is a Class 1 $\mathcal{C}_{n-1}^*$ matrix, of which there are $(n-1)!2^{n-1}$. We will use counting to see that every Class 1 $\mathcal{C}_{n-1}^*$ matrix is possible in this given "shell". The first row has $2n+1$ entries, but since the middle column contain only 0's, there are $2n$ positions that the 1 can occupy. Thus we are dividing the Class 1 $\mathcal{C}_n^*$ matrices into $2n$ groups. Let G_i, $i \in \{1, 2, \ldots, n, n+2, n+3, \ldots, 2n+1\}$, represent the group in which all matrices have a 1 in the ith column in their first row. Based on the way that Class 1 matrices are defined, we know that $|G_i|$ is the same for all i. There are $n!2^n$ Class 1 $\mathcal{C}_n^*$ matrices. Then it follows that $|G_i| = (n-1)!2^{n-1}$ for all i. For any G_i, the matrices in the set are identical in the deleted rows and columns. But since each rank n Class 1 matrix is unique, each modified Class 1 $\mathcal{C}_{n-1}^*$ matrix is different. Thus we conclude that each G_i corresponds to the entire set of Class 1 $\mathcal{C}_{n-1}^*$ matrices.

Using the bijection between $\mathcal{C}_n^*$ and Γ_n^*, we can carry this idea over to the set of ice states that correspond to Class 1 $\mathcal{C}_n^*$ matrices; call this set Ψ_n^*. Let A be a Class 1 $\mathcal{C}_n^*$ matrix with a 1 in the ith column of its first row. Let $\psi \in \Psi_n^*$ be its corresponding ice state. Deleting the first and last row of A is equivalent to deleting the first and last row of ψ. Since ψ only captures the left half of A, deleting the column with the 1 in ψ is now problematic if $i \in \{n+2, n+3, \ldots, 2n+1\}$. By the definition of Class 1 matrices, we know that there is exactly one 1 in each of the first and the last row of A. Let j be the column number of the 1 in the last row. The half-turn symmetry of A requires that if the 1 in the first row appears in the second half of the first row, then the 1 in the last row appears in the first half of the row (and vice versa). More precisely, for $i \in \{n+2, n+3, \ldots, 2n+1\}$, it is the case that $j = 2n+1-i \in \{1, 2, \ldots, n\}$. Since ψ only captures the left half of A, there will be exactly one EW vertex in the first and last row of ψ. Thus, deleting the column with the 1 in the first row of A is equivalent to deleting the column with an EW vertex in the first or the last column of ψ. By extension, each G_i is in bijection with the set Ψ_{n-1}^*. For a specific column number $i \in \{1, 2, \ldots, n\}$, let g_j denote the weight of the deleted vertices if the EW vertex is in the first row and g_{2n+1-j} if the EW vertex is in the last row. Then a set G_j contributes $g_j M(n-1)$ to the $M(n)$ coefficient if $j \in \{1, 2, \ldots, n\}$, $g_{2n+1-j} M(n-1)$ if $j \in \{n+2, n+3, \ldots, 2n+1\}$.

Define $f : \mathbb{Z}/2\mathbb{Z} \times [n] \to \mathbb{Z}[a, \bar{a}]$ as

$$f(z, j) = \begin{cases} \sigma\left(a^2\right) \bar{a}^{4n-4j+2} & \text{when } x = 0 \\ \sigma\left(a^2\right) a^{4n-4j+2} & \text{when } x = 1 \end{cases},$$

where $0 \in \mathbb{Z}/2\mathbb{Z}$ tells us that the EW is in the first row, $1 \in \mathbb{Z}/2\mathbb{Z}$ means that the EW vertex is in the last row, and $j \in [n]$ gives the column number of the EW vertex. Our claim is that $f(0, j) = g_j$ and that $f(1, j) = g_{2n+1-j}$. To prove this, we will calculate $g(0, j)$ and $g(1, j)$ separately.

- Case 1: the EW vertex is in the jth column of the first row

  Refer to Figure 21 for a pictorial representation of Case 1.

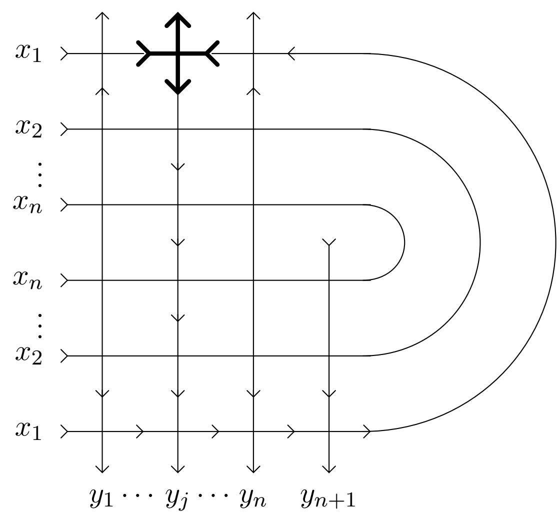


FIGURE 21. Ice model "Shell" in Case 1

Consider the first row. We know that there is an EW vertex in the jth column, which contributes a weight of $\sigma\left(a^2\right)$. There is only one EW vertex in the first and last row of the ice state. Recall that Class 1 matrices have no -1 entries, so the corresponding ice state has no NS vertices, which forces all other vertices in the first row to be zero vertices, which is a vertex with one inward edge along the horizontal line, and one inward edge along the vertical line. The boundary arrows at the top of the vertical lines forces all $n-1$ vertices to be southernly. The first vertex in the first row is westernly, which means all the $j-1$ vertices before the EW vertex are all westernly. The $n-(j+1)+1 = n-j$ after the EW vertex must all be easternly. Then row 1 has $j-1$ SW vertices and $n-j$ SE vertices. Thus, the first row contributes a total weight of $\sigma\left(a^2\right) a^{j-1} \left(-\bar{a}\right)^{n-j}$.

Consider the last row. Since the EW vertex is in the first row, we know that the last row consist entirely of zero vertices. There are $n+1$ vertices in the last row and the boundary arrows at the bottom of the vertical lines forces all $n+1$ to be northernly. Since the first vertex in the last row is westernly and there are no nonzero vertices in the row, it follows that all other vertices are westernly as well. Then row $2n$ has $n+1$ NW vertices and contributes a total weight of $(-\bar{a})^{n+1}$.

Consider the jth column. We have already accounted for the EW vertex in the first row and the NW vertex in the last row, so we are only looking at rows 2 through $2n-1$. Having an EW in the first row and NW in the last row forces all vertices in column j to be northernly. In any particular row, the vertex will be westernly if there is no EW vertex to its left, and it will be easternly if there is an EW vertex to its left. By construction, every column (except for the middle column) in a Class 1 matrix has exactly one 1. Then there must be an EW vertex in every column (except for the last column) of the ice state. Thus there are $j-1$ EW vertices to the left of column j. We already know that they are not in row 1 nor $2n$, so they must be in rows 2 through $2n-1$. Then there are $j-1$ NE vertices and $(2n-2)-(j-1) = 2n-j-1$ NW vertices along the middle of the jth column. We conclude that the middle $2n-2$ vertices in the jth column contribute weight of $a^{j-1}(-\bar{a})^{n-j-1}$.

When the EW vertex is in the first row, we have that

$$
\begin{aligned}
g_j = & \underbrace{\left[\sigma\left(a^2\right) a^{j-1}(-\bar{a})^{n-j}\right]}_{\text{first row}} \underbrace{\left[(-\bar{a})^{n+1}\right]}_{\text{last row}} \underbrace{\left[a^{j-1}(-\bar{a})^{2n-j-1}\right]}_{\text{middle of } j\text{th column}} \\
= & \sigma\left(a^2\right) a^{(j-1)+(j-1)}(-\bar{a})^{(n-j)+(n+1)+(2n-j-1)} \\
= & \sigma\left(a^2\right) a^{2j-2}(-\bar{a})^{4n-2j} \\
= & \sigma\left(a^2\right) \bar{a}^{-2j+2} \bar{a}^{4n-2j} \\
= & \sigma\left(a^2\right) \bar{a}^{4n-4j+2}
\end{aligned}
$$

where the second to last equality follows because $a = \bar{a}^{-1}$ and $4n-2j$ is even.

- Case 2: the EW vertex is in the jth column of the last row

  Refer to Figure 22 for a pictorial representation of Case 2.

  Consider the first row. Since the EW vertex is in the last row, we know that the first row consist entirely of zero vertices. There are n vertices in the first row and the boundary arrows at the top of the vertical lines forces all n vertices to be southernly. Since the first

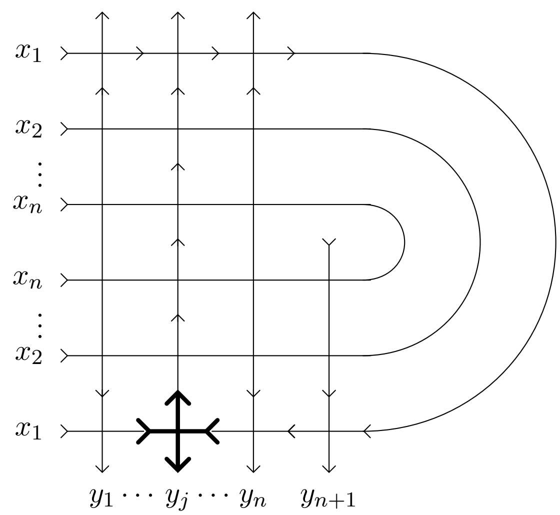


FIGURE 22. "Shell" in Case 2

vertex in the first row is westernly and there are no nonzero vertices in the row, it follows that all other vertices are westernly as well. Then row 1 has n SW vertices and contributes a total weight of a^n.

Consider the last row. We know that it has an EW vertex that contributes a weight of $\sigma\left(a^2\right)$. There is only one EW vertex in the first and last row of the ice state. Recall that Class 1 matrices have no -1 entries, so the corresponding ice state has no NS vertices, which forces all other vertices in the last row to be zero vertices. The boundary arrows at the bottom of the vertical lines forces all n vertices to be northernly. The first vertex in the last row is westernly, which means all the $j-1$ vertices before the EW vertex are all westernly. The $(n+1)-(j+1)+1 =$ after the EW vertex must all be easternly. Then row $2n$ has $j-1$ NW vertices and $n-j+1$ NE vertices. Thus, the last row contributes a total weight of $\sigma\left(a^2\right) a^{n-j+1} \left(-\bar{a}\right)^{j-1}$.

Consider the jth column. We have already accounted for the SW vertex in the first row and the EW vertex in the last row, so we are only looking at rows 2 through $2n-1$. Having a SW in the first row and EW in the last row forces all vertices in column j to be southernly. In any particular row, the vertex will be westernly if there is no EW vertex to its left, and it will be easternly if there is an EW vertex to its left. By construction, every column (except for the middle column) in a Class 1 matrix has exactly one 1. Then there must be an EW vertex in every column (except for the last column) of the ice state.

Thus there are $j-1$ EW vertices to the left of column j. We already know that they are not in row 1 nor $2n$, so they must be in rows 2 through $2n-1$. Then there are $j-1$ SE vertices and $(2n-2)-(j-1)=2n-j-1$ SW vertices along the middle of the jth column. We conclude that the middle $2n-2$ vertices in the jth column contribute weight of $a^{2n-j-1}\left(-\bar{a}\right)^{j-1}$. When the EW vertex is in the last row, we have that

$$\begin{aligned}
g_{2n+1-j} = &\underbrace{\left[a^n\right]}_{\text{first row}}\underbrace{\left[\sigma\left(a^2\right)a^{n-j+1}\left(-\bar{a}\right)^{j-1}\right]}_{\text{last row}}\underbrace{\left[a^{2n-j-1}\left(-\bar{a}\right)^{j-1}\right]}_{\text{middle of }j\text{th column}}\\
&=\sigma\left(a^2\right)a^{(n)+(n-j-1)+(2n-j-1)}\left(-\bar{a}\right)^{(j-1)+(j-1)}\\
&=\sigma\left(a^2\right)a^{4n-2j}\left(-\bar{a}\right)^{2j-2}\\
&=\sigma\left(a^2\right)a^{4n-2j}\bar{a}^{2j-2} && (2j-2\text{ is even})\\
&=\sigma\left(a^2\right)a^{4n-2j}a^{-2j+2} && (\bar{a}=a^{-1})\\
&=\sigma\left(a^2\right)a^{4n-4j+2}.
\end{aligned}$$

Therefore, we know that given the row and column number of the EW vertex in the first/last row, the function f gives the weight of the deleted vertices. We showed earlier in the proof that each such set of matrices contribute $gM(n-1)$ to the $M(n)$ coefficient, which is equal to $fM(n-1)$. Thus, if we sum over all possible row and column positions, the resulting polynomial would equal $M(n)$. This is equivalent to summing over all possible values for $z\in\mathbb{Z}/2\mathbb{Z}$ and $j\in[n]$. Then, we have that

$$\begin{aligned}
M(n) &= \left[\sum_{j=1}^{n}\sum_{z=0}^{1}f(z,j)\right]M(n-1)\\
&=\left[\sum_{j=1}^{n}\sigma\left(a^2\right)\left(a^{4n-4j+2}+\bar{a}^{4n-4j+2}\right)\right]M(n-1)\\
&=\left[\sigma\left(a^2\right)\sum_{j=1}^{n}\left(a^{4n-4j+2}+\bar{a}^{4n-4j+2}\right)\right]M(n-1)\\
&=\left[\sigma\left(a^2\right)\left(a^{4n-2}+a^{4n-6}+\cdots+a^2+\bar{a}^2+\cdots+\bar{a}^{4n-6}+\bar{a}^{4n-2}\right)\right]M(n-1)\\
&=\sigma(a^{4n})M(n-1) && \text{(Lemma 5.2).}
\end{aligned}$$

★

6. $S(n; x_n, y_{n+1})$ Coefficient

We now consider the terms of $\tilde{Z}_{\mathcal{C}_n^*}(\vec{x},\vec{y})$ that give the maximum degree in all the variables x_i except x_n and do not contain any of the variables y_i except y_{n+1}. Singling out these terms, we write

$$\tilde{Z}_{\mathcal{C}_n^*}(\vec{x},\vec{y}) = \left[\prod_{i=1}^{n-1} x_i^{4n}\right] S(n; x_n, y_{n+1}) + \dots.$$

The above definition does not exclude terms that have maximal degree in x_n.

Recall that $\tilde{Z}_{\mathcal{C}_n^*}(\vec{x},\vec{y})$ is equal to the original partition function, $Z_{\mathcal{C}_n^*}(\vec{x},\vec{y})$, scaled by a factor of $y_{n+1}^n\left[\prod_{i=1}^n x_i^{2n}y_i^{2n-1}\right]$. The scale factor was chosen so that there would be no negative exponents on any of the variables. In other words, n is largest exponent possible for $\bar{y}_{n+1}$, $2n$ is largest exponent possible for $\bar{x}_i$ and $2n-1$ is the largest exponent possible for $\bar{y}_i$, $1 \le i \le n$. In order for a term in $\tilde{Z}_{\mathcal{C}_n^*}(\vec{x},\vec{y})$ to have maximal degree in the x_i's, its counterpart in $Z_{\mathcal{C}_n^*}(\vec{x},\vec{y})$ must also have maximal degree in the x_i's (in this case the maximum possible power is $2n$ instead of $4n$). In order for a term in $\tilde{Z}_{\mathcal{C}_n^*}(\vec{x},\vec{y})$ to not contain any of the variables y_i, $1 \le i \le n$, its counterpart in $Z_{\mathcal{C}_n^*}(\vec{x},\vec{y})$ must have maximal degree in $\bar{y}_i$. Since the terms in question can contain y_{n+1}, their counterparts in $Z_{\mathcal{C}_n^*}(\vec{x},\vec{y})$ can have any degree of y_{n+1} or $\bar{y}_{n+1}$. To summarize, the terms we are interested in have counterparts in $Z_{\mathcal{C}_n^*}(\vec{x},\vec{y})$ with maximal degree $2n$ in all variables x_i except x_n and contains $\bar{y}_i^{2n-1}$ for $1 \le i \le n$. Singling out these terms, we write

$$Z_{\mathcal{C}_n^*}(\vec{x},\vec{y}) = \left[\prod_{i=1}^{n-1} x_i^{2n} \prod_{i=1}^{n} \bar{y}_i^{2n-1}\right] S(n; x_n, y_{n+1}) + \dots.$$

Only vertices that translate into 0 in matrix form can contribute a positive power of x_i. The definition of $\mathcal{C}_n^*$ matrices dictates that each row have at least one 1 since each row must sum to 1. We are looking for the terms that have maximal degrees for $x_1, x_2, \dots, x_{n-1}$, so we only want to consider the states that have only one 1 in rows $1, 2, \dots, n-1, n+2, \dots, 2n-2, 2n$ (the rows associated with $x_1, x_2, \dots, x_{n-1}$). There are two possibilities for rows n and $n+1$:

(1) each row has only one nonzero entry, a 1, or

(2) at least one row has one other nonzero entry.

Matrices belonging to the first case are precisely Class 1 matrices.

Because of the half-turn symmetry of the matrix, rows n and $n+1$ will have the same number of nonzero entries. Without loss of generality, assume that row $n+1$ has additional nonzero entry(ies).

The restriction on the sum of the entries in a row forces the least number of nonzero entries that row $n+1$ can have to be three: two 1 and one -1. Each row is sign-alternating, which dictates that the nonzero entries show up in the following order: 1, -1, and 1. Recall that we want maximum degree in the variables $\bar{y}_i$ for $1 \leq i \leq n$, which requires that we minimize the number of nonzero entries in the first n entries in each row. Thus, we want the -1 and 1 to be in entries $n+1, \ldots, 2n+1$ of row $n+1$. Notice that having both -1 and 1 in the last n entries of row $n+1$ would force row n have have two nonzero entries in its first n entries, which goes against maximizing the degree of $\bar{y}_i$ for $1 \leq i \leq n$. It must be that the -1 is in the $n+1$st position while the second 1 is in the last n positions. Half-turn symmetry tells us that row n also has a 1 in its first n entries, a -1 in its $n+1$st entry, and another 1 in its last n entries. Matrices belonging to the second case are precisely Class 2 matrices.

The weights of Class 1 matrices contain two distinct terms satisfying the condition of maximal degree in x_i except x_n and containing no y_i except for y_{n+1}. The first term is formed by taking the portion of nonzero vertices' weights that contain x_i. This gives a term with maximal degree in all x_i (which automatically gives maximal degree in all $\bar{y}_i$), including x_n. Once scaled by the scale factor, this term becomes $\prod_{i=1}^{n} x_i^{4n}$ in $\tilde{Z}_{\mathcal{C}_n^*}(\vec{x}, \vec{y})$. The only vertex in the ice model whose weight does not affect the degree maximality of $x_1, x_2, \ldots, x_{n-1}$ is vertex $(n+1, n+1)$, which has value $x_n\bar{y}_{n+1}$. By definition of Class 1 matrix, this vertex corresponds to a 0 in the matrix. This vertex will either have weight $\sigma(ax_n\bar{y}_{n+1}) = ax_n\bar{y}_{n+1} - \bar{a}\bar{x}_n y_{n+1}$ (if it has SW or NE configuration) or $\sigma(a\bar{x}_n y_{n+1}) = a\bar{x}_n y_{n+1} - \bar{a}x_n\bar{y}_{n+1}$ (if it has SE or NW configuration). Either way, its weight has a portion containing $x_n\bar{y}_{n+1}$ and a portion containing $\bar{x}_n y_{n+1}$. In the term described above, we only considered the portion with $x_n\bar{y}_{n+1}$. If we were to consider the portion with $\bar{x}_n y_{n+1}$, the resulting term would be $y_{n+1}^2 x_n^{4n-2} \prod_{i=1}^{n-1} x_i^{4n}$.

Class 2 matrices only contribute a term of the form $y_{n+1}^2 x_n^{4n-2} \prod_{i=1}^{n-1} x_i^{4n}$. Let $N(n)$ denote coefficient of the $y_{n+1}^2 x_n^{4n-2} \prod_{i=1}^{n-1} x_i^{4n}$ contributed by Class 1 matrices. Let $E(n)$ denote the coefficient of $y_{n+1}^2 x_n^{4n-2} \prod_{i=1}^{n-1} x_i^{4n}$ contributed by Class 2 matrices. Since Class 1 and Class 2 matrices are the only $\mathcal{C}_n^*$ matrices that contribute to the $S(n; x_n, y_{n+1})$ coefficient, we conclude that

$$S(n; x_n, y_{n+1}) = M(n)x_n^{4n} + [N(n) + E(n)]x_n^{4n-2}y_{n+1}^2.$$

Lemma 6.1.

Let $s \in S_n$ and $x_i \in \mathbb{Z}/2\mathbb{Z}$ for $i = 1, 2, \ldots, n-1$. The Class 1 matrices associated with $s \times x_1 \times x_2 \times \cdots \times x_{n-1} \times 0$ and $s \times x_1 \times x_2 \times \cdots \times x_{n-1} \times 1$ contribute the same polynomial to the $N(n)$ coefficient.

Proof. For clarity, let us refer to the matrix associated with $s \times x_1 \times x_2 \times \cdots \times x_{n-1} \times 0$ as matrix A. We will call its corresponding ice state α. Let us refer to the matrix associated with $s \times x_1 \times x_2 \times \cdots \times x_{n-1} \times 1$ as matrix B and call its corresponding ice state β. By the construction of Class 1 matrices, A and B are identical except in rows n and $n+1$, which means that α and β are identical except in rows n and $n+1$. See Figure 23 for α and β. If we can verify that the two rows in α and β contribute the same factors to $N(n)$, then it follows that A and B contribute the same polynomial to $N(n)$.

Consider matrix A. Recall that rank n Class 1 matrices are built using rank n permutation matrices and flipping (or not flipping) the rows in the permutation matrix. In matrix A, the last row of the permutation matrix is not flipped, and so there is a 1 in its nth row. By the definition of permutation matrix, we know that the 1 appears in position $(n, s^{-1}(n))$. Because the permutation is put on the left hand side of the Class 1 matrix, we know that the 1 appears in the first half of row n, i.e. $s^{-1}(n) \in \{1, 2, \ldots, n\}$. From this we can conclude that the vertex in the position $(n, s^{-1}(n))$ in α has EW configuration. We know that all other entries in row n of A are 0. This means that for $i \neq s^{-1}(n)$, vertex (n, i) in α is a zero vertex. The $s^{-1}(n) - 1$ vertices to the left of the EW vertex are westernly while the $n - (s^{-1}(n) + 1) + 1 = n - s^{-1}(n)$ vertices to the right of the EW vertex are easternly. By construction, row $n+1$ in A consists only of 0's, and so row $n+1$ in α consists entirely of westernly vertices, as forced by the boundary arrow on the left of the horizontal lines. More specifically, vertices $(n+1, s^{-1}(n))$ and $(n+1, n+1)$ have NW configuration.

Consider matrix B. Matrix B is built from the same permutation matrix as A, except that the nth row of the permutation matrix is "flipped". This means that there is a 1 in position $(n+1, s^{-1}(n))$ of B. It follows that the vertex in position $(n+1, s^{-1}(n))$ of β has EW configuration. Row $n+1$ of B has exactly one 1 with all other entries 0, which means that for $i \neq s^{-1}(n)$, vertex $(n+1, i)$ in β is a zero vertex. The boundary edge on the left of the horizontal lines force the $s^{-1}(n)$ vertices to the left of the EW vertex to be westernly. The $(n+1) - (s^{-1}(n) + 1) + 1 = n - s^{-1}(n) + 1$ vertices to the right of the EW vertex must be easternly. Specifically, we know that the vertex in position

$(n+1, n+1)$ has NE configuration. By construction, row n of B consists only of 0's, and so all vertices in row n of β are zero vertices. The boundary condition forces all of them to be westernly. Specifically, we know that the vertex in the $(n, s^{-1}(n))$ position has SW configuration.

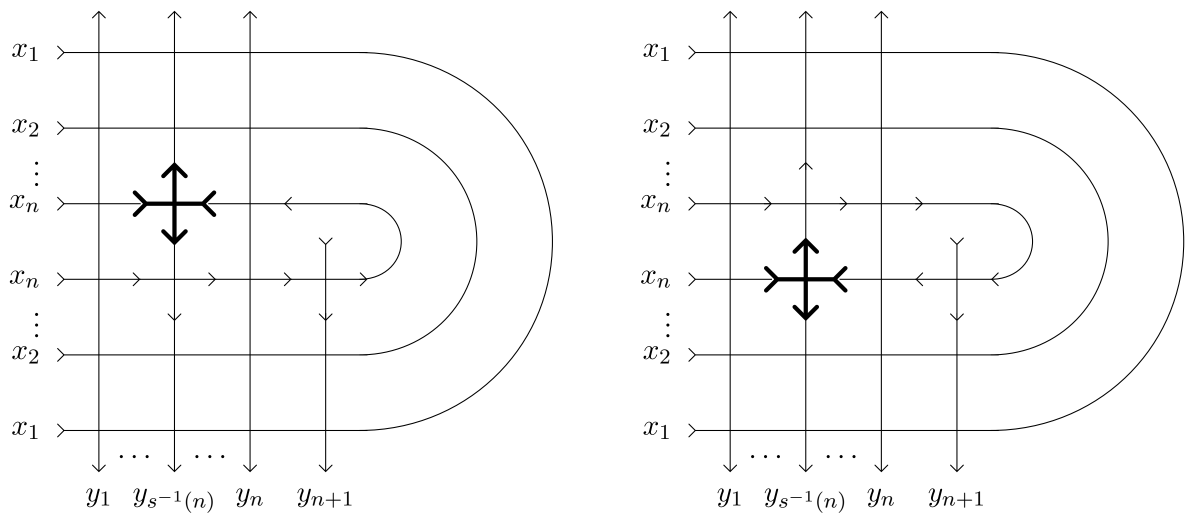


FIGURE 23. Ice states α and β

For $k \in \{1, 2, \ldots, n\} \setminus \{s^{-1}(n)\}$, vertices (n, k) and $(n+1, k)$ are zero vertices. Since the two vertices are vertically adjacent, they must have the same vertical direction, i.e. vertex (n, k) is northernly if and only if vertex $(n+1, k)$ is northernly. Taking all of the information we have so far, we conclude that for $k \in \{1, 2, \ldots, n\} \setminus \{s^{-1}(n)\}$, vertex (n, k) in α has the same configuration is vertex $(n+1, k)$ in β and vice versa.

There are now 3 vertices in each matrix that have not been taken into account: $(n, s^{-1}(n))$, $(n+1, s^{-1}(n))$, and $(n+1, n+1)$. We know that in α, the vertices have configurations EW, NW, and NW, respectively. For β, the vertices have configurations SW, EW, and NE, respectively. We know that vertices $(n, s^{-1}(n))$ in α and $(n+1, s^{-1}(n))$ in β both contribute a factor of $\sigma\left(a^2\right)$ to their respective matrix's weight. Since we are specifically looking at these matrices' contribution to $N(n)$, for each vertex in the ice model, we only consider the portion of the vertex's weight containing x_i. Then for α, the two unaccounted vertices contribute a factor of $(-\bar{a})(a) = -1$ to its lead coefficient. In β, its two unaccounted vertices contribute a factor of $(a)(-\bar{a}) = -1$ to its weight.

We have now shown, vertex by vertex, that the two matrices contribute the same polynomial to the $N(n)$ coefficient. ★

Lemma 6.2.

The coefficient $N(n)$ *is given by the formula*

$$N(n) = -2\sigma\left(a^2\right)\prod_{i=2}^{n}\sigma(a^{4i}).$$

Proof. Let A be a Class 1 $\mathcal{C}_n^*$ matrix. Let $\alpha \in \Gamma_n^*$ be its corresponding ice state. We know that $wt(\alpha)$ contributes both to $M(n)$ and $N(n)$. The differentiating factor in the ice state's contribution to $M(n)$ versus $N(n)$ is in vertex $(n+1, n+1)$. When considering the matrix's contribution to $M(n)$, which is the coefficient of the term in which all x_i's are of maximal power $(x_1^{4n}x_2^{4n}\cdots x_n^{4n})$, we take the portion of the vertex's weight involving x_n. When considering the matrix's contribution to $N(n)$, which is the coefficient of the term $x_1^{4n}x_2^{4n}\cdots x_{n-1}^{4n}x_n^{4n-2}y_{n+1}^2$, we take the portion of the vertex's weight involving y_{n+1}.

The $\mathcal{C}_n^*$ boundary condition forces vertex $(n+1, n+1)$ to be northernly, see Figure 24. If vertex $(n+1, n+1)$ has NE configuration, it contributes a toward $M(n)$ and $-\bar{a}$ toward $N(n)$. Thus, the matrix's contribution to $M(n)$ is $-a^2$ times its contribution to $N(n)$. If vertex $(n+1, n+1)$ has NW configuration, it contributes $-\bar{a}$ toward $M(n)$ and a toward $N(n)$. Thus, the matrix's contribution to $M(n)$ is $-\bar{a}^2$ times its contribution to $N(n)$. Vertex $(n+1, n+1)$ is of NE configuration when there is a 1 in row $n+1$, which is precisely when the last flip value is 1. Vertex $(n+1, n+1)$ is of NW configuration when the $n+1$th row is all 0, which is precisely when the last flip value is 0.

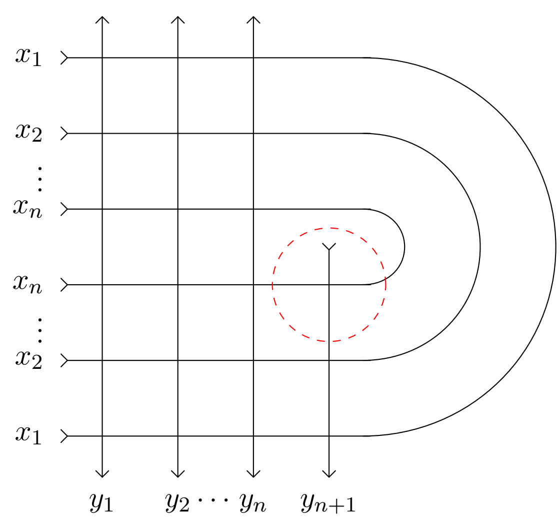


FIGURE 24. Vertex in position $(n+1, n+1)$

Thus Class 1 matrices can be divided into two groups based on the last flip value. By Lemma 6.1, we know that each group contributes the same polynomial to $N(n)$. We then conclude that each group contributes $\frac{1}{2}N(n)$. Using the information above, we conclude that

$$\begin{aligned} M(n) &= (-a^2)\left(\frac{1}{2}N(n)\right) + (-\bar{a}^2)\left(\frac{1}{2}N(n)\right) \\ &= -\frac{1}{2}(a^2+\bar{a}^2)N(n). \end{aligned}$$

Rewriting the equation gives:

$$\begin{aligned} N(n) &= -2M(n)(a^2+\bar{a}^2)^{-1} \\ &= -2\left[\prod_{i=1}^{n}\sigma(a^{4i})\right](a^2+\bar{a}^2)^{-1} && \text{(Lemma 5.5)} \\ &= -2\left[\sigma(a^4)\prod_{i=2}^{n}\sigma(a^{4i})\right](a^2+\bar{a}^2)^{-1} \\ &= -2\left[\sigma\left(a^2\right)(a^2+\bar{a}^2)\prod_{i=2}^{n}\sigma(a^{4i})\right](a^2+\bar{a}^2)^{-1} \\ &= -2\sigma\left(a^2\right)\prod_{i=2}^{n}\sigma(a^{4i}). && \bigstar \end{aligned}$$

7. Partition Function of $\mathcal{D}_n$ Matrices

The $\mathcal{D}_n$ matrices are similar to $\mathcal{C}_n^*$ matrices in that they have half-turn symmetry, and rows and columns are sign-alternating. The differences are that $\mathcal{D}_n$ matrices have one less column than row, and a different restriction is put on its middle column.

Definition 7.1.

Let $\mathcal{D}_n$ be the set of all $2n \times (2n-1)$ matrices $A = (a_{i,j})_{1\le i\le 2n, 1\le j\le 2n-1}$ satisfying the following conditions:

(1) For all $1 \le p \le 2n$, row p is sign-alternating with $\sum_{j=1}^{2n-1} a_{p,j} = 1$.

(2) For all $1 \le q \le 2n-1$, except for $q = n$, column q is sign-alternating with $\sum_{i=1}^{2n} a_{i,q} = 1$.

(3) The vector $(a_{1,n}, \ldots, a_{n,n})$ is sign-alternating and $\sum_{i=1}^{n} a_{i,n} = 1$.

(4) $a_{i,j} = a_{2n+1-i,2n-j}$.

Condition (1) and (2) tell us that all rows and all columns (except for the middle column) are sign-alternating and that their entries sum to 1. This forces the rows and columns with nonzero entries to begin and end with 1. Condition (3) says that the top half of the middle column (column n) is sign-alternating and must sum to 1. Thus, the middle column always has nonzero entries, that begin and end with 1. Condition (4) is the half-turn symmetry condition.

Lemma 7.2.

There exists a bijection between $\mathcal{D}_n$ and ice states satisfying the $\mathcal{D}_n$ boundary condition, shown in Figure 25. Let Δ_n be the set of ice states corresponding to the set of $\mathcal{D}_n$ matrices.

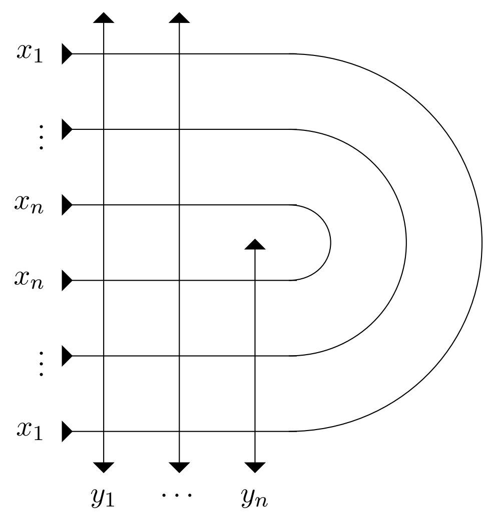


FIGURE 25. $\mathcal{D}_n$ boundary condition

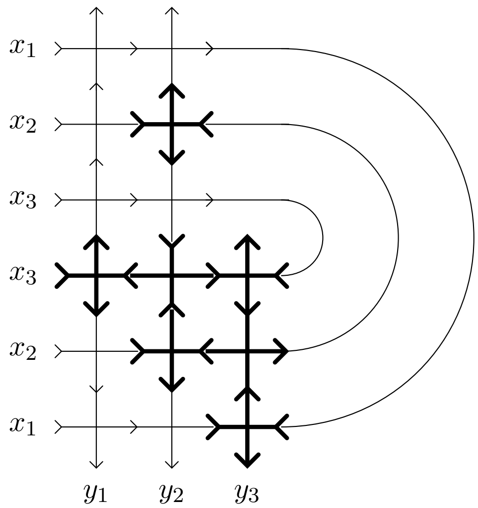
As in the $\mathcal{C}_n^*$ boundary condition, this ice model only records the lower left half of the entries in a $2n \times (2n-1)$ matrix, but the half-turn symmetric nature of the $\mathcal{D}_n$ matrices dictates what the other entries must be. Translating the vertices in an ice model with the $\mathcal{D}_n$ boundary condition into numbers using the chart given in Figure 4, we get the lower left half of a $\mathcal{D}_n$ matrix (the pseudo-matrix in the middle of Figure 26).

Definition 7.3 ($Z_{\mathcal{D}_n}(\vec{x},\vec{y})$)**.**

Let $Z_{\mathcal{D}_n}(\vec{x},\vec{y})$ denote the partition function of $\mathcal{D}_n$ matrices. It is equal to the sum over the weights of all elements of Δ_n, i.e.

$$Z_{\mathcal{D}_n}(\vec{x},\vec{y}) = \sum_{\delta \in \Delta_n} wt(\delta).$$

As in the case with $Z_{\mathcal{C}_n^*}(\vec{x},\vec{y})$, we want a modified version of $Z_{\mathcal{D}_n}(\vec{x},\vec{y})$ that has no negative powers on any of its variables.



$$\Leftrightarrow \begin{pmatrix} 0 & 0 & \\ 0 & 1 & \\ 0 & 0 & \\ 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \Leftrightarrow \begin{bmatrix} 0 & 0 & \mathbf{1} & 0 & 0 \\ 0 & \mathbf{1} & \mathbf{-1} & \mathbf{1} & 0 \\ 0 & 0 & \mathbf{1} & \mathbf{-1} & \mathbf{1} \\ \mathbf{1} & \mathbf{-1} & \mathbf{1} & 0 & 0 \\ 0 & \mathbf{1} & \mathbf{-1} & \mathbf{1} & 0 \\ 0 & 0 & \mathbf{1} & 0 & 0 \end{bmatrix}$$

FIGURE 26. Illustration of the bijection between the $\mathcal{D}_n$ boundary condition and $\mathcal{D}_n$ matrices

Definition 7.4 ($\tilde{Z}_{\mathcal{D}_n}(\vec{x},\vec{y})$)**.**

Define a modified version of the partition function as

$$\tilde{Z}_{\mathcal{D}_n}(\vec{x},\vec{y}) = y_n^{n-1}\left[\prod_{i=1}^{n} x_i^{2n-2}\prod_{j=1}^{n-1} y_i^{2n-1}\right] Z_{\mathcal{D}_n}(\vec{x},\vec{y}).$$

Theorem 7.5.

The modified partition function $\tilde{Z}_{\mathcal{D}_n}(\vec{x},\vec{y})$ has the following properties:

(a) $\tilde{Z}_{\mathcal{D}_n}(\vec{x},\vec{y})$ is symmetric in the variables $x_1,\ldots,x_n$.

(b) $\tilde{Z}_{\mathcal{D}_n}(\vec{x},\vec{y})$ is symmetric in the variables $y_1,\ldots,y_{n-1}$.

(c) $Z_{\mathcal{D}_n}(\vec{x},\vec{y})$ is invariant under the simultaneous replacement $x_i \mapsto \bar{x}_i$, $y_i \mapsto \bar{y}_i$, for $i \in \{1,2,\ldots,n\}$. In other words, $Z_{\mathcal{D}_n}(\vec{x},\vec{y}) = Z_{\mathcal{D}_n}(\bar{\vec{x}},\bar{\vec{y}})$ where $\bar{\vec{x}} = (\bar{x},\bar{x}_2,\ldots,\bar{x}_n)$ and $\bar{\vec{y}} = (\bar{y},\bar{y}_2,\ldots,\bar{y}_n)$. For the modified partition function one has

$$\tilde{Z}_{\mathcal{D}_n}(\bar{\vec{x}},\bar{\vec{y}}) = \bar{y}_n^{2n-2}\left[\prod_{i=1}^{n} \bar{x}_i^{4n-4}\prod_{j=1}^{n-1} \bar{y}_j^{4n-2}\right] \tilde{Z}_{\mathcal{D}_n}(\vec{x},\vec{y}).$$

(d) $\tilde{Z}_{\mathcal{D}_n}(\vec{x},\vec{y})$ is invariant under the replacement $a \mapsto -a$.

(e) $\tilde{Z}_{\mathcal{D}_n}(\vec{x},\vec{y})$ *is a homogeneous polynomial in the variables* x_i *and* y_i *of total degree* $4n^2-4n$. *For each fixed* $i=1,\ldots,n$, *it is a polynomial in* x_i^2 *of degree* $2n-2$. *For each* $j=1,\ldots,n-1$, *it is a polynomial in* y_i^2 *of degree* $2n-1$. *It is a polynomial in* y_n^2 *of degree* $n-1$.

(f) If $y_1 = ax_1$, *then*

$$\frac{Z_{\mathcal{D}_n}(\vec{x},\vec{y})}{Z_{\mathcal{D}_{n-1}}(\vec{x}',\vec{y}')} = \sigma\left(a^2\right)\sigma(a\bar{x_1}y_1)\sigma(a\bar{x_1}y_n)\left[\prod_{j=2}^{n-1}\sigma(a\bar{x_1}y_j)^2\right]\left[\prod_{k=2}^{n}\sigma(a\bar{x_k}y_1)^2\right],$$

$$\frac{\tilde{Z}_{\mathcal{D}_n}(\vec{x},\vec{y})}{\tilde{Z}_{\mathcal{D}_{n-1}}(\vec{x}',\vec{y}')} = \sigma\left(a^2\right)x_1^{2n-2}y_1^{2n-2}(a\bar{x}_1y_1^2-\bar{a}x_1)(a\bar{x}_1y_n^2-\bar{a}x_1)$$
$$\left[\prod_{k=2}^{n-1}\left(a\bar{x}_1y_k^2-\bar{a}x_1\right)^2\right]\left[\prod_{l=2}^{n}\left(ay_1-\bar{a}x_l^2\bar{y}_1\right)^2\right]$$

where $\vec{x}' = \vec{x}\setminus\{x_1\} = \{x_2,x_3,\ldots,x_n\}$ *and* $\vec{y}' = \vec{y}\setminus\{y_1\} = \{y_2,y_3,\ldots,y_n\}$.

Proof.

(a) Identical to the proof of Lemma 3.11.

(b) Identical to the proof of Lemma 3.16.

(c) Analogous to the proof of Lemma 3.17.

(d) Because the parameter a is not present in the scale factor of $Z_{\mathcal{D}_n}(\vec{x},\vec{y})$ that results in $\tilde{Z}_{\mathcal{D}_n}(\vec{x},\vec{y})$, if we can prove that $Z_{\mathcal{D}_n}(\vec{x},\vec{y})$ is invariant under the replacement $a\mapsto -a$, then it follows that its modified version is also. We will show that for each matrix $A\in\mathcal{D}_n$, the weight of the corresponding ice state $\alpha\in\Delta_n$ is a polynomial in a^2. We know that the weight of α has the following form (see part (e) of current proof):

$$wt(\alpha) = \left(a^2-\bar{a}^2\right)^{b_\alpha}(a\bar{x}_iy_j-\bar{a}\bar{x}_iy_j)^{c_\alpha}(a\bar{x}_iy_j-\bar{a}x_i\bar{y}_j)^{d_\alpha}.$$

The factors of $\left(a^2-\bar{a}^2\right)$ are contributed by vertices of configuration EW or NS. Since this factor is itself a polynomial in a^2, $(a^2-\bar{a}^2)^{b_\alpha}$ will be a polynomial in a^2 independent of the value of b_α (the number of EW and NS vertices in α). The factors $(a\bar{x}_iy_j-\bar{a}\bar{x}_iy_j)$ and $(a\bar{x}_iy_j-\bar{a}x_i\bar{y}_j)$ come from the remaining four vertex configurations, all of which correspond to an entry of 0 in A. If we can show that $c_\alpha+d_\alpha$ is even, then it follows that the weight of α is a polynomial in a^2. This is equivalent to proving that there are an even number of 0's in the half of A that is encoded by α. For ease of notation, we will refer to this pseudo-matrix as A'.

We know that A' has $n-1$ columns with $2n$ entries and a half column with n entries. By definition of $\mathcal{D}_n$ matrix, each of the first $n-1$ columns in A sums to 1. This means that in each column, there must be at least one 1 and that there is one more 1 than -1. Therefore, we can conclude that p_i, the number of nonzero vertices in column i, is odd for $1 \leq i \leq n-1$. Thus, column i in A' has $2n - p_i$ zeros. We know that the half-column must sum to 1, which means there is at least one 1 and that there is one more 1 than -1. From this we know that q, the number of nonzero vertices in the half column of A', is odd. The half column contains $n-q$ zero entries. Putting this information together, there are $\left(\sum_{i=1}^{n-1} 2n - p_i\right) + (n-q)$ zero entries in A'. Regardless of the parity of n, we know that $2n$ is even. And since p_i is odd, the difference $2n - p_i$ is odd for all i. We now break the proof up into two cases based on the parity of n:

(1) n is even

Then $\left(\sum_{i=1}^{n-1} 2n - p_i\right)$ is the sum of an odd number of odd numbers, so it is odd. And $n-q$ is the difference between an even number and an odd number, so it is odd as well. The sum of two odd numbers is even, so there are an even number of zeros in A'.

(2) n is odd

The expression $\left(\sum_{i=1}^{n-1} 2n - p_i\right)$ is an sum of an even number of odd numbers, so it is even. And $n-q$ is the difference between two odd numbers, so it is even as well. The sum of two even numbers is even, so there are an even number of zeros in A'.

(e) Let us first consider $Z_{\mathcal{D}_n}(\vec{x}, \vec{y})$. Let A be a $\mathcal{D}_n$ matrix. Let α be the ice state in Δ_n corresponding to A. Consulting Figure 10, we see that each vertex in α contributes either $\sigma\left(a^2\right) = a^2 - \bar{a}^2$, $\sigma(ax_i\bar{y}_j) = a\bar{x}_i y_j - \bar{a}\bar{x}_i y_j$, or $\sigma(a\bar{x}_i y_j) = a\bar{x}_i y_j - \bar{a}x_i\bar{y}_j$ to the weight of α. Notice that every term (the terms being $a^2, -\bar{a}^2, ax_i\bar{y}_j, \bar{a}\bar{x}_i y_j$, etc.) in every factor has degree 0 with respect to $\{x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n\}$. The weight of α is the product of the weights of its vertices, so it must have the form

$$wt(\alpha) = (a^2 - \bar{a}^2)^{b_\alpha}(a\bar{x}_i y_j - \bar{a}\bar{x}_i y_j)^{c_\alpha}(a\bar{x}_i y_j - \bar{a}x_i\bar{y}_j)^{d_\alpha}.$$

Then the weight of α is a sum of products of terms with degree zero. Thus each term in $wt(\alpha)$ has degree zero. Since $Z_{\mathcal{D}_n}(\vec{x}, \vec{y})$ is a sum over the weights of all $\alpha \in \Delta_n$, we know that it has the form

$$\sum_{\alpha \in \Delta_n} wt(\alpha) = \sum_{\alpha \in \Delta_n} (a^2 - \bar{a}^2)^{b_\alpha} (a\bar{x}_i y_j - \bar{a}\bar{x}_i y_j)^{c_\alpha} (a\bar{x}_i y_j - \bar{a} x_i \bar{y}_j)^{d_\alpha}.$$

Then each term in $Z_{\mathcal{D}_n}(\vec{x}, \vec{y})$ has degree 0.

We know that $\tilde{Z}_{\mathcal{C}_n^*}(\vec{x}, \vec{y})$ is equal to $y_n^{n-1} \left[\prod_{i=1}^n x_i^{2n-2} \prod_{j=1}^{n-1} y_i^{2n-1} \right]$ times $Z_{\mathcal{D}_n}(\vec{x}, \vec{y})$. We've proven above that each term in $Z_{\mathcal{D}_n}(\vec{x}, \vec{y})$ has degree zero, thus each term in $\tilde{Z}_{\mathcal{D}_n}(\vec{x}, \vec{y})$ has degree equal to the degree of $y_n^{n-1} \left[\prod_{i=1}^n x_i^{2n-2} \prod_{j=1}^{n-1} y_i^{2n-1} \right]$. The degree of $y_n^{n-1} \left[\prod_{i=1}^n x_i^{2n-2} \prod_{j=1}^{n-1} y_i^{2n-1} \right]$ is

$$(n-1) + (2n-2)(n) + (2n-1)(n-1) = n - 1 + 2n^2 - 2n + 2n^2 - n - 2n + 1 = 4n^2 - 4n.$$

By the definition of homogeneous, we conclude that $\tilde{Z}_{\mathcal{D}_n}(\vec{x}, \vec{y})$ is homogeneous of total degree $4n^2 - 4n$.

Recall that the scale factor $y_n^{n-1} \left[\prod_{i=1}^n x_i^{2n-2} \prod_{j=1}^{n-1} y_i^{2n-1} \right]$ was chosen based on the maximum possible degree for the inverse of each variable. The reasoning we used to prove that the maximum possible degree of $\bar{x}_i$ in $Z_{\mathcal{D}_n}(\vec{x}, \vec{y})$ must be $2n-2$ can be used to conclude that the maximum possible degree of x_i must be $2n-2$ as well. Similarly, the maximum degree of y_i, $1 \leq i \leq n-1$, and the maximum degree of y_{n+1} are $2n-1$ and $n-1$, respectively. In $\tilde{Z}_{\mathcal{D}_n}(\vec{x}, \vec{y})$, then, the maximum degree of x_i is $(2n-2)+(2n-2) = 4n-4$, y_i is $(2n-1)+(2n-1) = 4n-2$, and y_n is $(n-1) + (n-1) = 2n-2$. Thus, $\tilde{Z}_{\mathcal{D}_n}(\vec{x}, \vec{y})$is a polynomial in x_i^2 of degree $2n-2$, a polynomial in y_i^2 of degree $2n-1$, and is a polynomial in y_n^2 of degree $n-1$.

(f) Let $y_1 = ax_1$. Let A be a $\mathcal{D}_n$ matrix. Let $\alpha \in \Delta_n$ be the ice state corresponding to A. If the top left corner entry of A were a 0, then the boundary condition on α dictates that the top left corner vertex of α be of SW configuration. The weight of this vertex would be $\sigma(ax_1\bar{y}_1) = ax_1(\bar{a}\bar{x}_1) - \bar{a}\bar{x}_1(ax_1) = 1 - 1 = 0$. Then the weight of α would also be 0, and so α would not contribute to $Z_{\mathcal{D}_n}(\vec{x}, \vec{y})$.

Thus, we only need to consider the matrices in $\mathcal{D}_n$ with a nonzero entry in the top left corner. Let A be such a matrix. By definition of $\mathcal{D}_n$, the top row of A must be sign-alternating, which forces the first nonzero entry in the row to be 1. In the case when $y_1 = ax_1$, the only matrices in $\mathcal{D}_n$ that contribute nontrivially to $Z_{\mathcal{D}_n}(\vec{x}, \vec{y})$ are the ones with a 1 in their top left corner. This means that their corresponding ice states must an EW vertex in the upper left corner. If

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & & & & & & 0 \\ 0 & & & & & & 0 \\ \vdots & & & & & & \vdots \\ 0 & & & & & & 0 \\ 0 & & & & & & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix} \quad \Leftrightarrow$$

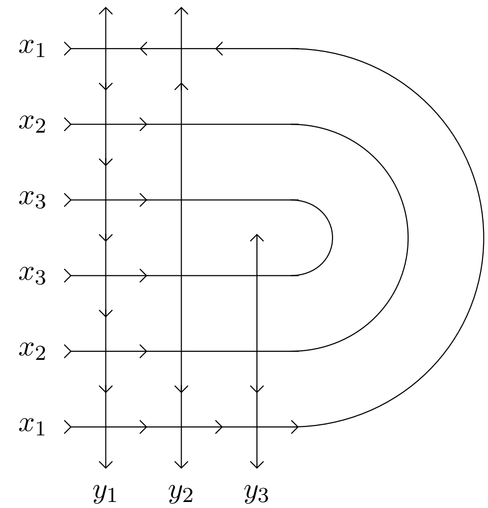


FIGURE 27. "Boundary" of A and "outer shell" of α
With the exception of the vertex in the upper left corner, all vertices in the top row have SE configuration, all vertices in the last row have NW configuration, and all vertices in the first column have NW configuration.

the first row in A were to have other nonzero entries, then the nonzero entry after the 1 must be -1, for the row to be sign-alternating. But we also know that each column in A is sign-alternating, and so the first nonzero element in any column must be 1. Therefore, we conclude that all other entries in the first row of A are 0's. Similarly, there cannot be other nonzero elements in the first column of A. Thus it is a column beginning with 1 and followed by 0's. By the half-turn symmetry of the $\mathcal{D}_n$ matrices, we know that the last row of A is all 0 except for the last entry, which has value 1 and that the last column of A is all 0 except for the last entry. The matrix in Figure 27 shows the form that A must have. The information we know about the "boundary" of A, combined with the boundary condition α, allow us to pinpoint the configurations that occur on the "outer shell" of α, shown in the ice state in Figure 27.

Notice that the "interior" of α is precisely the boundary condition for an ice state corresponding to a $\mathcal{D}_{n-1}$ matrix. Then Δ_{n-1} gives all possible ways of filling out the "interior" of α. We know that all $\mathcal{D}_n$ matrices that contribute nontrivially to $Z_{\mathcal{D}_n}(\vec{x}, \vec{y})$ have the same "boundary" as A, which means that their corresponding ice states have the"outer shell" as α. This tells us that $Z_{\mathcal{D}_n}(\vec{x}, \vec{y})$ is equal to the weight of the "outer shell" of α multiplied by $Z_{\mathcal{D}_{n-1}}(\vec{x}', \vec{y}')$. In other words the ratio $\frac{Z_{\mathcal{D}_n}(\vec{x},\vec{y})}{Z_{\mathcal{D}_{n-1}}(\vec{x}',\vec{y}')}$ is equal to the weight of the "outer shell".

Since we know the vertex configurations of the vertices in the "outer shell", we can compute its weight by breaking the shell into four parts:

- Upper left corner has value 1: $\sigma\left(a^2\right)$
- All other vertices in the first row have SE configuration: $\prod_{i=2}^{n-1} \sigma(a\bar{x}_1 y_i)$
- All vertices in the bottom row have NW configuration: $\prod_{j=1}^{n} \sigma(a\bar{x}_1 y_j)$
- The vertices in the left column that have not already be accounted for have NW configuration: $\prod_{k=2}^{n} \sigma(a\bar{x}_k y_1)^2$

Putting everything together, we have:

$$\begin{aligned}
&\frac{\mathcal{D}(n;\vec{x},\vec{y})}{\mathcal{D}(n-1;\vec{x}',\vec{y}')} \\
=&\sigma\left(a^2\right)\left[\prod_{i=2}^{n-1}\sigma(a\bar{x}_1 y_i)\right]\left[\prod_{j=1}^{n}\sigma(a\bar{x}_1 y_j)\right]\left[\prod_{k=2}^{n}\sigma(a\bar{x}_k y_1)^2\right] \\
=&\sigma\left(a^2\right)\sigma(a\bar{x}_1 y_1)\sigma(a\bar{x}_1 y_n)\left[\prod_{j=2}^{n-1}\sigma(a\bar{x}_1 y_j)^2\right]\left[\prod_{k=2}^{n}\sigma(a\bar{x}_k y_1)^2\right]
\end{aligned} \tag{7.6}$$

Using Equation (7.6) and Definition 7.4, we calculate:

$$
\begin{aligned}
&\frac{\tilde{Z}_{\mathcal{D}_n}(\vec{x},\vec{y})}{\tilde{Z}_{\mathcal{D}_{n-1}}(\vec{x}',\vec{y}')}\\
=&\frac{Z_{\mathcal{D}_n}(\vec{x},\vec{y})}{Z_{\mathcal{D}_{n-1}}(\vec{x}',\vec{y}')} \times \frac{y_n^{n-1}\left[\prod_{i=1}^{n} x_i^{2n-2} \prod_{j=1}^{n-1} y_j^{2n-1}\right]}{y_n^{n-2}\left[\prod_{k=2}^{n} x_k^{2n-4} \prod_{l=2}^{n-1} y_l^{2n-3}\right]}\\
=&\frac{Z_{\mathcal{D}_n}(\vec{x},\vec{y})}{Z_{\mathcal{D}_{n-1}}(\vec{x}',\vec{y}')} \times x_1^{2n-2} y_1^{2n-1} y_n \left[\prod_{k=2}^{n} x_k^2 \prod_{l=2}^{n-1} y_l^2\right]\\
=&\sigma\left(a^2\right)\sigma(a\bar{x}_1 y_1)\sigma(a\bar{x}_1 y_n)\left[\prod_{j=2}^{n-1}\sigma(a\bar{x}_1 y_j)^2\right]\left[\prod_{k=2}^{n}\sigma(a\bar{x}_k y_1)^2\right] x_1^{2n-2} y_1^{2n-1} y_n \left[\prod_{k=2}^{n} x_k^2 \prod_{l=2}^{n-1} y_l^2\right]\\
=&\sigma(a^2) x_1^{2n-2} y_1^{2n-2} [\sigma(a\bar{x}_1 y_1) y_1][\sigma(a\bar{x}_1 y_n) y_n]\left[\prod_{k=2}^{n-1}\sigma(a\bar{x}_1 y_k)^2 y_k^2\right]\left[\prod_{l=2}^{n}\sigma(a\bar{x}_l y_1)^2 x_l^2\right]\\
=&\sigma(a^2) x_1^{2n-2} y_1^{2n-2} [(a\bar{x}_1 y_1 - \bar{a} x_1 \bar{y}_1) y_1][(a\bar{x}_1 y_n - \bar{a} x_1 \bar{y}_n) y_n]\times\\
&\left[\prod_{k=2}^{n-1}((a\bar{x}_1 y_k - \bar{a} x_1 \bar{y}_k) y_k)^2\right]\left[\prod_{l=2}^{n}((a\bar{x}_l y_1 - \bar{a} x_l \bar{y}_1) x_l)^2\right]\\
=&\sigma\left(a^2\right) x_1^{2n-2} y_1^{2n-2} (a\bar{x}_1 y_1^2 - \bar{a} x_1)(a\bar{x}_1 y_n^2 - \bar{a} x_1)\left[\prod_{k=2}^{n-1}\left(a\bar{x}_1 y_k^2 - \bar{a} x_1\right)^2\right]\left[\prod_{l=2}^{n}\left(a y_1 - \bar{a} x_l^2 \bar{y}_1\right)^2\right].
\end{aligned}
$$

★

Given more time, our goal is to find an explicit formula for the lead coefficient analogous to $S(n; x_n, y_{n+1})$ for $\tilde{Z}_{\mathcal{D}_n}(\vec{x},\vec{y})$. Our analysis suggests that $\tilde{Z}_{\mathcal{D}_n}(\vec{x},\vec{y})$ and $\tilde{Z}_{\mathcal{C}_n^*}(\vec{x},\vec{y})$ can be expressed in terms of each other. If true, it would mean that the enumeration of $\mathcal{C}_\backslash^*$ matrices and the enumeration of $\mathcal{D}_n$ are can be given in terms of each other. The ultimate goal would be to relate $\tilde{Z}_{\mathcal{D}_n}(\vec{x},\vec{y})$ to the partition functions whose explicit formulas are known in order to find an explicit formula for $\tilde{Z}_{\mathcal{D}_n}(\vec{x},\vec{y})$. Only then can be evaluate the partition function at special values to enumerate the $\mathcal{D}_n$ matrices.

References

[1] David Bressoud and James Propp. How the alternating sign matrix conjecture was solved. *Notices of the AMS*, 46(6):637–646, 1999.

[2] Noam Elkies, Greg Kuperberg, Michael Larsen, and James Propp. Alternating sign matrices and domino tilings. *Journal of Algebraic Combinatorics*, 2:111–132, 1991.

[3] Greg Kuperberg. Symmetry classes of alternating-sign matrices under one roof. *Annals of Mathematics*, 156:835–866, 2002.

[4] A. V. Razumov and Yu. G. Stroganov. Enumerations of half-turn symmetric alternating-sign matrices of odd order. *Theoretical and Mathematical Physics*, 148(3):1174–1198, 2006.

[5] Doron Zeilberger. Proof of the alternating sign matrix conjecture. *Electronic Journal of Combinatorics*, 3(2), 1996.