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Alf Dolich

Julia F. Knight

Karen Lange
klange2@wellesley.edu

David Marker

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Representing Scott Sets in Algebraic Settings*

Alf Dolich

Kingsborough Community College

Julia F. Knight

University of Notre Dame

Karen Lange[†]

Wellesley College

David Marker

University of Illinois at Chicago

1 Introduction

Recall that $S \subseteq 2^\omega$ is called a *Scott set* if and only if:

i) S is a Turing ideal, i.e., if $x, y \in S$ and $z \leq_T x \oplus y$, then $z \in S$, where $x \oplus y$ is the disjoint union of x and y ;

ii) If $T \subseteq 2^{<\omega}$ is an infinite tree computable in some element of S , then there is $f \in S$ an infinite path through T .

Scott sets first arose in the study of completions of Peano arithmetic (PA) and models of PA. Scott [9] shows that the countable Scott sets are exactly the families of sets “representable” in a completion of PA. If \mathcal{M} is a nonstandard model of Peano arithmetic and $a \in \mathcal{M}$, let

$$r(a) = \{n \in \omega : \mathcal{M} \models p_n | a\}$$

where p_0, p_1, \dots is an increasing enumeration of the standard primes. The *standard system* of \mathcal{M}

$$SS(\mathcal{M}) = \{r(a) : a \in \mathcal{M}\}$$

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is a Scott set. A longstanding and vexing problem in the study of models of arithmetic is whether every Scott set arises as the standard system of a model of Peano arithmetic. The best result is from Knight and Nadel [6].

Proposition 1.1 *If S is a Scott set and $|S| \leq \aleph_1$, then there is a model of Peano Arithmetic with standard system S .*

Thus the Scott set problem has a positive solution if the Continuum Hypothesis is true, but the question remains open without additional assumptions. Later in this section, we sketch a proof of Proposition 1.1.

Scott sets also are important when studying recursively saturated structures. We assume that we are working in a computable language \mathcal{L} . We fix a Gödel coding of \mathcal{L} and say that a set of \mathcal{L} -formulas is in S if the corresponding set of Gödel codes is in S .

Let T be a complete \mathcal{L} -theory. The following ideas were introduced in [5], [10] and [7].

Definition 1.2 Let $S \subseteq 2^\omega$. We say that a model \mathcal{M} of T is *S -saturated* if:

- i) every type $p \in S_n(\emptyset)$ realized in \mathcal{M} is computable in some element of S ;
- ii) if $p(x, \bar{y}) \in S_{n+1}(\emptyset)$ is computable in some element of S , $\bar{a} \in M^n$ and $p(x, \bar{a})$ is finitely satisfiable in \mathcal{M} , then $p(x, \bar{a})$ is realized in \mathcal{M} .

If a model is S -saturated for some $S \subseteq 2^\omega$, then the model is certainly recursively saturated.

Proposition 1.3 *If $\mathcal{M} \models T$ is recursively saturated, then \mathcal{M} is S -saturated for some Scott set S .*

If the theory T has limited coding power, then we can say little about S . For example, an algebraically closed field of infinite transcendence degree will be S -saturated for every Scott set S . On the other hand, the associated Scott set is unique for many natural examples, such as Peano arithmetic, divisible ordered abelian groups, real closed fields, \mathbb{Z} -groups (models of $\text{Th}(\mathbb{Z}, +)$) and Presburger arithmetic (models of $\text{Th}(\mathbb{Z}, +, <)$).

Definition 1.4 We say that a theory T is *effectively perfect* if there is a tree $(\phi_\sigma : \sigma \in 2^{<\omega})$ of formulas in n -free variables computable in T such that:

- i) $T + \exists \bar{v} \phi_\sigma(\bar{v})$ is consistent for all σ ;
- ii) if $\sigma \subset \tau$, then $T \models \phi_\tau(\bar{v}) \rightarrow \phi_\sigma(\bar{v})$;
- iii) $\phi_{\sigma \smallfrown 0}(\bar{v}) \wedge \phi_{\sigma \smallfrown 1}(\bar{v})$ is inconsistent with T for all σ .

Proposition 1.5 *If T is effectively perfect, then every recursively saturated model of T is S -saturated for a unique Scott set S .*

The theories we will be considering are all effectively perfect. For Peano arithmetic, Presburger arithmetic and \mathbb{Z} -groups we can use the formulas $p_n|v$ to find such a tree. For real closed fields we can use $q < v$ for $q \in \mathbb{Q}$ and for ordered divisible abelian groups we can use the binary formulas $mv < nw$ for $m, n \in \mathbb{Z}$.

We now sketch a proof of Proposition 1.1. We first note that for models \mathcal{M} of Peano arithmetic, \mathcal{M} is recursively saturated if and only if \mathcal{M} is S -saturated where S is the standard system of \mathcal{M} . (For more details see [4]).

Lemma 1.6 *If S is a countable Scott set and $T \in S$ is a completion of Peano arithmetic, then there is an S -saturated model of T .*

Proof Sketch Build \mathcal{M} by a Henkin construction. At any stage, we will have a finite tuple \bar{a} and will be committed to $\text{tp}(\bar{a})$ the complete type of \bar{a} , where $T \subseteq \text{tp}(\bar{a}) \in S$. At alternating stages, we either witness an existential quantifier or realize a type $p(v, \bar{a}) \in S$, using the join property of Scott sets to compute $p(v, \bar{x}) \cup \text{tp}(\bar{a})$, and using the tree property to find completions. \square

Lemma 1.7 *Suppose $S_0 \subset S_1$ are countable Scott sets, $T \in S_0$ is a completion of Peano arithmetic, and \mathcal{M}_0 and \mathcal{M}_1 are countable recursively saturated models of T , where S_i is the standard system of \mathcal{M}_i . Then there is an elementary embedding of \mathcal{M}_0 into \mathcal{M}_1 .*

Proof Sketch Let a_0, a_1, \dots be a list of the elements of \mathcal{M}_0 . Suppose we have a partial elementary map $(a_0, \dots, a_n) \mapsto (b_0, \dots, b_n)$. If $\text{tp}(a_{n+1}, a_0, \dots, a_n) = p(v, a_0, \dots, a_n)$, there is $b \in \mathcal{M}_1$ realizing $p(v, b_0, \dots, b_n)$, and we can extend the embedding. *Box*

We can now prove Proposition 1.1. Suppose $|S| = \aleph_1$ and S is the union of an ω_1 -chain of countable Scott sets

$$S_0 \subseteq S_1 \subseteq \dots \subseteq S_\alpha \subseteq \dots$$

where $S_\alpha = \bigcup_{\beta < \alpha} S_\beta$ when α is a limit ordinal. We can build an elementary chain $(\mathcal{M}_\alpha : \alpha < \omega_1)$ where \mathcal{M}_α is recursively saturated with standard

system S_α . Then $\bigcup_{\alpha < \omega_1} \mathcal{M}_\alpha$ is recursively saturated with standard system S .

While we have nothing new to say about the Scott set problem for Peano arithmetic, we show that the analogous problem for recursively saturated models has a positive solution in some related algebraic settings.

For divisible ordered abelian groups, this follows easily from an unpublished result of Harnik and Ressayre. Let $(G, +, <)$ be a divisible ordered abelian group. Define an equivalence relation on G by $g \equiv h$ if and only if there is a natural number n such that $|g| < n|h|$ and $|h| < n|g|$. Let $\Gamma = \{|g|/\equiv : g \in G\}$, the set of equivalence classes of positive elements. The ordering of G induces an ordering on Γ . Suppose S is a Scott set and k_S is the set of real numbers computable in some element of S (where we identify a real with its cut in the rationals). It is easy to see that k_S is a real closed field.

Theorem 1.8 (Harnik–Ressayre) *A divisible ordered abelian group G is S -saturated if and only if Γ is a dense linear order without endpoints and each equivalence class under \equiv is isomorphic to the ordered additive group of k_S .*

A complete proof is given in [3].

Corollary 1.9 *For any Scott set S , there is an S -saturated divisible ordered abelian group.*

Proof Let G be the set of functions $f : \mathbb{Q} \rightarrow k_S$ such that $\{q \in \mathbb{Q} : f(q) \neq 0\}$ is finite. We add elements of G coordinatewise and order G lexicographically. By the Harnik–Ressayre Theorem, G is S -saturated. \square

2 Real Closed Fields

In [2] D’Aquino, Knight and Starchenko show that if \mathcal{M} is a nonstandard model of Peano arithmetic with standard system S , then the real closure of the fraction field of \mathcal{M} is an S -saturated real closed field. Thus, it is natural to ask whether we can find an S -saturated real closed field for every Scott set S .

Theorem 2.1 *For any Scott set S , there is an S -saturated real closed field.*

The value group of an S -saturated real closed field will be an S -saturated divisible ordered abelian group. Thus, Corollary 1.9 will also follow from Theorem 2.1.

Theorem 2.1 is a simple induction using the following Lemma.

Lemma 2.2 *Let S be a Scott set. Let K be a real closed field such that every type realized in K is in S . Suppose $p(v, \bar{w})$ is a set of formulas in S , $\bar{a} \in K$, and $p(v, \bar{a})$ is finitely satisfiable in K . Then we can realize $p(v, \bar{a})$ by a (possibly new) element b such that every type realized in $K(b)^{\text{rcl}}$ is in S .*

Proof The set of formulas $p(v, \bar{a}) \cup \text{tp}(\bar{a})$ is a consistent partial type in S , and, hence, has a completion in S . Thus, without loss of generality, we may assume $p(v, \bar{a})$ is a complete type. If $p(v, \bar{a})$ is realized in K , then there is nothing to do. If $p(v, \bar{a})$ is not realized in K then it determines a cut in the ordering of $\mathbb{Q}(\bar{a})^{\text{rcl}}$ that is not realized in K , and, hence, by o-minimality, it determines a unique type over K . Let b realize $p(v, \bar{a})$ and let $\bar{c} \in K$. We need to show that $\text{tp}(b, \bar{c})$ is in S .

How do we decide whether $K(b)^{\text{rcl}} \models \phi(b, \bar{c})$? By o-minimality, $\phi(v, \bar{c})$ defines a finite union of points and intervals with endpoints in $\mathbb{Q}(\bar{c})^{\text{rcl}}$. Since $b \notin K$, b is neither one of the distinguished points nor an end point of one of the intervals. There are \emptyset -definable Skolem functions f and g such that:

- i) $f(\bar{a}) < v < g(\bar{a}) \in \text{tp}(b/\bar{a})$;
- ii) $f(\bar{a}) < v < g(\bar{a}) \rightarrow \phi(v, \bar{c})$ or $f(\bar{a}) < v < g(\bar{a}) \rightarrow \neg\phi(v, \bar{c})$;

Given $\text{tp}(b/\bar{a})$ and $\text{tp}(\bar{a}, \bar{c})$ we can computably search and find the decomposition of $\phi(v, \bar{c})$ and f and g as above. We can then decide whether $\phi(b, \bar{c})$ holds. Since S is closed under join and Turing reducibility, $\text{tp}(b, \bar{c})$ is in S . \square

The above argument works for any o-minimal theory $T \in S$.

Every real closed field K has a natural valuation for which the valuation ring is

$$\mathcal{O} = \{x : |x| < n \text{ for some } n \in \mathbb{N}\}.$$

If K is recursively saturated, then the value group is a recursively saturated divisible ordered abelian group. It is natural to ask if every recursively saturated divisible ordered abelian group arises this way.

D'Aquino, Kuhlmann and Lange [3] gave a valuation-theoretic characterization of recursively saturated real closed fields. In the following argument we assume familiarity with their results.¹

¹This is essentially our original proof of Theorem 2.1.

Proposition 2.3 *Let G be a recursively saturated divisible ordered abelian group. There is a recursively saturated real closed field with value group G .*

Proof Sketch Let S be the Scott set of G . Start with the field

$$k_S(t^g : g \in G)^{\text{rcf}}.$$

This is a real closed field with residue field k_S , value group G and all types recursive in S . Given K a real closed field K with value group G and all types recursive in S and suppose we have $\bar{a} \in K$ and (f_0, f_1, \dots) a sequence of Skolem functions recursive in S such that $(f_0(\bar{a}), f_1(\bar{a}), \dots)$ is pseudo-Cauchy. If the sequence has no pseudo-limit in K it determines a unique type over K . Adding a realization b does not change the value group or residue field. As above, every type realized in $K(b)^{\text{rcf}}$ is in S . We can iterate this construction to build the desired real closed field. \square

3 Presburger Arithmetic

In [6] Knight and Nadel proved that for every Scott set S there is an S -saturated \mathbb{Z} -group, i.e., an S -saturated model of $\text{Th}(\mathbb{Z}, +)$. They asked whether the same is true for the theory of $(\mathbb{Z}, +, <)$. This is Presburger arithmetic, which we denote Pr . We answer this question in the affirmative.

Theorem 3.1 *For every Scott set S , there is an S -saturated model of Presburger arithmetic.*

We will consider Presburger arithmetic in the language that includes constants for 0 and 1 and unary predicates $P_n(v)$ for $n = 2, 3, \dots$ that hold if n divides v . We can eliminate quantifiers in this language and the resulting structure is quasi-o-minimal; i.e., any formula $\phi(v, \bar{a})$ defines a finite Boolean combination of \emptyset -definable sets and intervals with endpoints in $\text{dcl}(\bar{a}) \cup \{\pm\infty\}$. We will use this in the following form. (See, for example [8] §3.1 for quantifier elimination and [1] for quasi-o-minimality.)

Lemma 3.2 *i) Any formula $\phi(v, \bar{a})$ is equivalent over $\text{tp}(\bar{a})$ to a Boolean combination of formulas of the form $v \equiv m \pmod{n}$, $v = \alpha$, $v < \beta$ where α, β are in the definable closure of \bar{a} and $m, n \in \mathcal{N}$.
ii) $\text{tp}(b, \bar{a})$ is determined by:*

- $\text{tp}(\bar{a})$;
- the sequence $b \bmod 2, b \bmod 3, b \bmod 4, \dots$;
- the cut of b in the definable closure of \bar{a} .

We obtain Theorem 3.1 by an iterated construction using the following lemma.

Lemma 3.3 *Let S be a Scott set. Let $G \models \text{Pr}$ such that every type realized in G is computable in S . Suppose $\bar{a} \in G$ and $p(v, \bar{w})$ is a complete type in S such that $p(v, \bar{a})$ is finitely satisfiable. Then there is $H \supseteq G$ such that $H \models \text{Pr}$, such that $p(v, \bar{a})$ is realized in G and every type realized in H is in S .*

Proof If $p(v, \bar{a})$ is realized in G , then there is nothing to do, so we assume $p(v, \bar{a})$ is not realized in G . Let $p^-(v, \bar{a})$ be the partial type describing the cut of v over the definable closure of \bar{a} , i.e., p^- consists of all formulas of the form $mv < \sum n_i a_i$ or $mv > \sum n_i a_i$ that are in p where $m, n_i \in \mathbb{Z}$.

Case 1: Suppose p^- is omitted in G .

Let b be any realization of p , and let H be the definable closure of $G \cup \{b\}$. It is enough to show that if $\bar{c} \in G$, then $\text{tp}(b, \bar{c})$ is in S . We will show that $\text{tp}(b, \bar{c})$ is recursive in $\text{tp}(b, \bar{a})$ and $\text{tp}(\bar{a}, \bar{c})$. Using only $\text{tp}(b, \bar{a})$, we can determine $b \bmod n$ for all n . Thus, we only need to consider formulas of the form $\alpha < v < \beta$ where $\alpha, \beta \in \text{dcl}(\bar{c})$. Since $p^-(v, \bar{a})$ is omitted, we can, as in the case of real closed fields, search to find $\gamma, \delta \in \text{dcl}(\bar{a})$ such that $\gamma < b < \delta$ and either $\alpha \leq \gamma < \delta \leq \beta$, $\delta < \alpha$ or $\gamma > \beta$. This can be done recursively in $\text{tp}(b, \bar{a})$ and $\text{tp}(\bar{a}, \bar{c})$. Thus, every type realized in H is in S .

Case 2: Suppose $b \in G$ realizes p^- .

Let \hat{b} be a realization of $p(v, \bar{a})$ and let $q_0(v)$ be the divisibility type of $\hat{b} - b$, i.e., if $\hat{b} \equiv m \bmod n$ and $b \equiv l \bmod n$ then " $v \equiv m - l \bmod n$ " $\in q_0$ for $l, m \in \mathbb{Z}$ and $n > 1$.

Let $q(v) \in S_1(G)$ be the unique type containing

- $q_0(v)$;
- $n < v$ for all $n \in \mathbb{Z}$;
- $v < g$ for all $g \in G$ such that $\mathbb{Z} < g$.

Let ϵ realize q and let H be the definable closure of $G \cup \{\epsilon\}$. Suppose $\alpha \in \text{dcl}(\bar{a})$ and $b < \alpha$. Since $G \models \text{Pr}$, we have $b + n < \alpha$ for all $n \in \mathbb{Z}$. Thus, $\epsilon < \alpha - b$ and $b + \epsilon < \alpha$. Similarly, if $\alpha < b$, then $\alpha < b + \epsilon$ and, thus, $b + \epsilon$ realizes $p^-(v, \bar{a})$. By the choice of q_0 , $b + \epsilon$ realizes $p(v, \bar{a})$.

Suppose $\bar{c} \in G$. It suffices to show that $\text{tp}(\epsilon, \bar{c})$ is in S . Without loss of generality, we may assume that $\bar{c} = (c_1, \dots, c_n)$ where all of the c_i are positive infinite and $1, c_1, \dots, c_n$ are linearly independent over \mathbb{Q} . We need to decide the signs of expressions of the form

$$r + s\epsilon + \sum_{i=1}^n t_i c_i$$

where $r, s, t_i \in \mathbb{Z}$. Such an expression is positive if and only if

- $\sum t_i c_i > 0$, or
- $\sum t_i c_i = 0$ and $q > 0$, or
- $\sum t_i c_i = s = 0$ and $r > 0$

This can be computed using $\text{tp}(\bar{c})$. Thus $\text{tp}(\epsilon, \bar{c})$ is recursive in q_0 and $\text{tp}(\bar{c})$. Hence, every type realized in H is in S . \square

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