Universal cycles for k-subsets of an n-set

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UNIVERSAL CYCLES FOR K-SUBSETS OF AN N-SET

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Abstract

Generalized from the classic de Bruijn sequence, a universal cycle is a compact cyclic list of information. Existence of universal cycles has been established for a variety of families of combinatorial structures. These results, by encoding each object within a combinatorial family as a length-$j$ word, employ a modified version of the de Bruijn graph to establish a correspondence between an Eulerian circuit and a universal cycle.

We explore the existence of universal cycles for $k$-subsets of the integers $\{1, 2, \ldots, n\}$. The fact that sets are unordered seems to prevent the use of the established encoding techniques used in proving existence. We explore this difficulty and introduce an intermediate step that may allow us to use the familiar encoding and correspondence to prove existence.

Moreover, mathematicians Persi Diaconis and Ron Graham hold that “the construction of universal cycles has proceeded by clever, hard, ad-hoc arguments” and that no general theory exists. Accordingly, our work pushes for a more general approach that can inform other universal cycle problems.
Acknowledgement

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Index of Notation

$\mathcal{P}(S)$  Power set of set $S$

$|S|$  Cardinality of set $S$.

$a|b$  Integer $a$ divides integer $b$

$n, k$  Integers such that $n > k$

$[n]$  Integers 1 through $n$, i.e. $\{1, \ldots, n\}$

$[n]_m$  Set of all $m$-subsets of $[n]$

$\binom{n}{m}$  Binomial coefficient

$\binom{n}{m, m, \ldots, m}$  Multinomial coefficient

$\lambda$  Element of $[\binom{n}{k-1}]$

$\delta$  Element of $[\binom{n}{k}]$

$k\Gamma_n$  Dimension $k$ cycle set of $n$

$\gamma$  Element of $[\binom{n}{k+1}]$ that is contained in a cycle set

$\Omega$  Set of size $\binom{n}{k}/(k + 1)$ with elements from $[\binom{n}{k+1}]$

$\omega$  Element of $[\binom{n}{k+1}]$ that is contained in a set $\Omega$

$G$  Graph

$H$  Hypergraph

$D$  Directed graph

$d^+(v)$  In degree of a vertex $v$

$d^-(v)$  Out degree of a vertex $v$

$T$  Transition graph

$(x_0x_1 \ldots x_k)$  Directed $k + 1$ cycle in a transition graph $T$

$\sim$  Equivalence relation

$[x_0x_1 \ldots x_k]$  Equivalence class of label $x_0x_1 \ldots x_k$

$\sigma$  Ordering of a set
1 Universal Cycles

Universal cycles address questions in the field of discrete mathematics. The idea of a universal cycle is to create a compact list of information within a string of characters. As a consequence, existence results for universal cycles maximize efficiency, particularly in the expanding areas of encryption and data storage [2].

Sanskrit memory wheels, mnemonic devices used to memorize grammatical patterns, are one of the oldest mathematical objects and are an example of a universal cycle. The modern notion of a universal cycle began much later with what we now call de Bruijn sequences.

1.1 de Bruijn Sequences

A de Bruijn Sequence is a cyclic sequence from a binary alphabet such that every possible length $k$ binary word appears as a consecutive subsequence exactly once.

Example 1.1. The cyclic sequence 00111010 is a valid de Bruijn sequence of all length three binary words.

The first three characters in the sequence form the word 001. If we shift our length three window by one, the characters in positions two through four form the word 011. By continuing this process we find 111, 110, 101, 010, 100, and 000. Note that the word 100 is formed by taking the last two digits in our sequence followed by the first digit. The word 000 is formed in a similar fashion by taking the last digit followed by the first two digits in our sequence. Thus, each length three binary word appears exactly once as a consecutive subsequence.

Dutch Mathematician Nicolaas Govert de Bruijn, for whom this type of universal cycle is named, proved certain results about what he called “circular arrangements of $2^n$ zeros and ones that show each $n$-letter word exactly once.” However, de Bruijn did not introduce the idea nor was he the first
mathematician to find the number of solutions. De Bruijn attributes the problem of how many de Bruijn sequences exist for length $n$ binary words to A. de Rivière [3]. De Rivière proposed the question in the French journal “L’Intermédiaire des Mathématiciens” in 1894. The problem was promptly solved by C. Flye Sainte-Marie in the same year.

While Sainte-Marie implicitly proved many of the results that appear in de Bruijn’s proof, the explicit graph theoretic approach that we will employ wasn’t developed until the 1940’s. This method was developed simultaneously and independently by de Bruijn and British mathematician Irving John Good. Again, the relevant object is named after de Bruijn.

A de Bruijn graph $D_k$ is a directed graph that represents the overlap between sequences of characters. (More general information about graphs can be found in Appendix A). Given an alphabet $A$, the de Bruijn graph $D_k$ of all length $k$ words with characters from $A$ has vertex and edge set

$$V(D_k) := \{\text{all length } k-1 \text{ words with characters from } A\}$$

$$E(D_k) := \{(x_1x_2\ldots x_{k-1}, y_1y_2\ldots y_{k-1}) : x_2 = y_1, x_3 = y_2, \ldots, x_{k-1} = y_{k-2}\}.$$  

In other words, if the last $k-2$ characters of word $v \in V(D_k)$ are identical to the first $k-2$ characters of word $w \in V(D_k)$, then there is a directed edge from $v$ into $w$ that represents their overlap. We label edge $(x_1x_2\ldots x_{k-1}, y_1y_2\ldots y_{k-1})$ with the word $x_1x_2\ldots x_{k-1}y_{k-1}$.

**Example 1.2.** The graph $D_3$ of all length 3 words on a binary alphabet.

![Diagram](image)

Notice that all length three binary words appear as an edge in $D_3$. This graph is Eulerian, meaning it contains a directed circuit that uses each edge exactly once. The de Bruijn sequence 00110101 corresponds to the Eulerian circuit 001, 011, 111, 110, 101, 010, 100, 000 in $D_3$.

More generally, these results hold for any de Bruijn graph $D_k$. 

2
**Theorem 1.3.** [9] The de Bruijn graph $D_k$ is Eulerian and the edge labels of any Eulerian circuit of $D_k$ form a de Bruijn sequence.

This correspondence between edge labels of an Eulerian circuit and length $k$ consecutive subsequences of a sequence is vital to our approach. When we generalize de Bruijn sequences to universal cycles, this idea will allow us to relate an Eulerian circuit to a universal cycle.

The fact that $D_k$ is Eulerian and that each length $k$ binary word appears as an edge in $D_k$ won’t prove useful for our purposes. Thus we omit that portion of the proof. A complete proof can be found in [9].

We need a way to identify an Eulerian graph $G$ without having to find an Eulerian circuit of $G$. We will also need the fact that the edge labels appearing in an Eulerian circuit of $D_k$ correspond to a de Bruijn sequence of those labels. The following lemma is a well-known result in graph theory.

**Lemma 1.4.** A directed graph $D$ is Eulerian if and only if $d^+(v) = d^-(v)$ for each vertex $v \in V(D)$ and if $D$ has at most one nontrivial component.

**Proof.** ($\Rightarrow$) Assume that $D$ is Eulerian.

We can write an Eulerian circuit of $D$ as a list of edges and vertices $v_0, e_1, v_2, \ldots, v_{k-1}, e_k, v_k = v_0$ such that $e_i \neq e_j$ for all $i \neq j$ and the edge $e_i$ has tail $v_{i-1}$ and head $v_i$. Every time a vertex $v_i$ appears in an Eulerian circuit of $D$ there is an edge $e_i$ going into $v_i$ and an edge $e_{i+1}$ coming out of $v_i$. Since the edges in an Eulerian circuit are distinct and every edge in $D$ appears in an Eulerian circuit of $D$ we can conclude that $d^+(v) = d^-(v)$ for all vertex $v \in V(D)$.

Let $v_i$ be the endpoint of edge $e$ and let $v_j$ be the endpoint of edge $e'$ for not necessarily distinct $e, e' \in E(D)$. The graph $D$ is Eulerian. Thus $e$ and $e'$ must be contained in an Eulerian circuit of $D$. Therefore there exists a directed path between any two vertices $v_i, v_j \in V(D)$, namely the path $v_i, e_{i+1}, \ldots, e_j, v_j$ contained in an Eulerian circuit. Thus $D$ has at most one nontrivial component.

($\Leftarrow$) Assume that directed graph $D$ satisfies the conditions $d^+(v) = d^-(v)$ for each vertex $v \in V(D)$ and $D$ has at most one nontrivial component. We'll proceed using induction on $|V(D)| = n$. 

3
Base Case: Let $n = 1$. Then $D$ consists of loops on a single vertex. Follow the edges in any order to construct an Eulerian circuit of $D$.

Induction step: Assume that the result is true for $1, 2, \ldots, k$. We’ll show that the result holds when $|V(D)| = k + 1$.

We begin by constructing a circuit $C$ in $D$. Start with a vertex $v_0 \in V(D)$ such that $d^+(v_0) \neq 0$. Select the next vertex $v_1$ from among the heads of edges with $v_0$ as a tail. Note that by starting at $v_0$, we used one of $v_o$’s out edges. Thus we have a total of $d^+(v_0)$ in edges and $d^-(v_0) - 1$ out edges remaining that are incident to $v_0$ and can be used in our circuit $C$.

Continue in this same manner by selecting a vertex $v_i$ from among the heads of edges with $v_{i-1}$ as a tail. Since $d^+(v_i) = d^-(v_i)$, there is a way to exit every vertex $v_i \neq v_0$ that we enter. Continue this process until you enter vertex $v_0$ and you cannot exit. We know that the process must terminate at $v_0$ because we can enter $d^+(v_0)$ times and only exit $d^-(v_0) - 1$ times since we started at $v_0$. Further, $C$ includes every edge incident to $v_0$ since otherwise the process would not have terminated.

If all the edges in $D$ have been traversed, then our circuit $C$ is an Eulerian circuit of $D$. Otherwise, consider the nontrivial components $H_j$ of $D - C$. Since every edge incident to $v_0$ appears in $C$, we know that $v_0 \notin V(D - C)$. Thus $|V(D - C)| \leq k$.

Thus $|V(H_j)| \leq |V(D - C)| \leq k$. Furthermore, when the edges in $C$ are removed from $D$, a vertex loses an in-degree for every out-degree lost. Thus, by the induction hypothesis, $H_j$ contains an Eulerian circuit.

We can now build an Eulerian circuit for $D$. Start at $v_0$ in our cycle $C$. Since $H_j$ and $C$ must have a vertex $v_j$ in common, when we reach $v_j$, detour on the Eulerian circuit of $H_j$ before continuing on to $v_{j+1}$. We will return to $v_0$ having traversed every edge in $D$ exactly once.

Thus $D$ is Eulerian by strong induction on $|V(D)| = n$. \hfill \Box

The fact that a directed graph $D$ that has at most one nontrivial component and that satisfies $d^+(v) = d^-(v)$ for all $v \in V(D)$ is Eulerian is fundamental to our process for proving the existence of universal cycles.

The final key idea from de Bruijn sequences is the correspondence between an Eulerian circuit and a de Bruijn sequence. In other words, the edge
labels on the edges of any Eulerian circuit of \( D_k \) form a de Bruijn sequence of those labels.

To see this, let \( C \) be an Eulerian circuit in \( D_k \). Let \( e = x_1 x_2 \ldots x_{k-1} x_k \) be an edge in \( C \). In \( C \), we traverse edge \( e \) into vertex \( v = x_2 x_3 \ldots x_{k-1} x_k \) and from \( v \) we traverse edge \( e' \). Since vertex \( v \) is the tail of edge \( e' \), we know that \( e' \) has label \( x_2 x_3 \ldots x_k y \) for some \( y \in \mathcal{A} \).

\[
\begin{array}{c}
x_1 x_2 \ldots x_{k-1} \xrightarrow{e} x_2 x_3 \ldots x_{k-1} x_k \xrightarrow{e'} x_3 \ldots x_k y
\end{array}
\]

To the left of the window where \( x_1 x_2 \ldots x_k \) will appear in our de Bruijn sequence we write \( y \) such that our sequence reads \( x_1 x_2 \ldots x_k y \).

\[
\begin{array}{c}
x_1 x_2 \\
\vdots \\
x_k \\
y
\end{array}
\]

Then \( e' = x_2 \ldots x_k y \) is the successive consecutive subsequence of our de Bruijn sequence.

### 1.2 Generalization

By definition, de Bruijn sequences contain every length \( k \) word on a given alphabet. A natural question arises: Given any set of length \( k \) words, is it possible to create a cyclic arrangement such that each word appears as a consecutive subsequence exactly once? In 1992, mathematicians Fan Chung, Persi Diaconis, and Ron Graham explored generalizations of de Bruijn sequences to examine this question.

Universal cycles generalize the idea of a de Bruijn sequence.

**Definition 1.5.** Let \( S \) be a set of length \( k \) strings from an alphabet \( \mathcal{A} \). A universal cycle of \( S \) is a cyclic sequence such that every string in \( S \) appears as a consecutive subsequence exactly once. Further, if a length \( k \) string is not an element of \( S \), then it does not appear in the universal cycle of \( S \).

**Example 1.6.** Let \( \mathcal{A} = \{a, b, c, d, e\} \). Consider the set \( S \) of words \( aac, abb, aca, aed, baa, bbc, bcc, bea, cae, cba, ccb, ceb, dce, ddc, eab, ebe, \) and \( eed \).
Notice that $S$ does not include every length three word on the alphabet $\mathcal{A}$. The words $aba$, $daa$, and $eda$ are among the words missing from $S$. The sequence $abbcbaacaeddebe$ is a valid universal cycle of $S$. To verify that this is in fact a valid universal cycle of $S$, notice that the sequence has 17 length three consecutive subsequences. Since $|S| = 17$ and each word in $S$ appears exactly once as a subsequence, we know that the universal cycle contains only words from $S$ as length three consecutive subsequences.

The sets $S$ that Chung, Diaconis, and Graham examined were not randomly generated. In their research, they examined universal cycles of various combinatorial structures by encoding objects as length $k$ words from an alphabet and building an Eulerian arc digraph that represents overlap between words. One of the objects they investigated is $k$-sets of an $n$-set.

### 1.3 $k$-Subsets of an $n$-Set

**Definition 1.7.** A universal cycle of $k$-sets of an $n$-set is a cycle such that all $k$-subsets of the integers $\{1, 2, \ldots, n\}$ appear exactly once as a consecutive subsequence.

**Example 1.8.** The list below includes all 2-subsets of $\{1, 2, 3, 4\}$.

- $\{1, 2\}$
- $\{1, 3\}$
- $\{1, 4\}$
- $\{2, 3\}$
- $\{2, 4\}$
- $\{3, 4\}$

Chung, Diaconis, and Graham wrote each size $k$ set as a length $k$ word. For instance, the set $\{1, 2\}$ can be written as the word 12 or 21. They then tried to construct a universal cycle of these length $k$ words.

**Example 1.9.** Let $n = 8$ and $k = 3$. We have a valid universal cycle

$$23415741684125634561278347856724678123572367135824581368.$$
Chung, Diaconis, and Graham noted that one of the difficulties of this particular problem is that each size 3 set can appear as any of 6 words in our universal cycle. For instance, the set \( \{4, 6, 8\} \) can appear as 468, 486, 648, 684, 846, or 864. In the example above, the set \( \{4, 6, 8\} \) appears as 684.

Because of this feature, Chung, et al. concluded that we cannot define a transition graph, a directed graph similar to a de Bruijn graph that allows us to establish a correspondence between an Eulerian circuit and a universal cycle. Despite this difficulty, Chung, et. al. found a necessary condition for the existence of a universal cycle of \( k \)-subsets of \( n \).

**Lemma 1.10.** *If the set of \( k \)-subsets of \( n \) has a universal cycle, then \( k \) divides \( \binom{n-1}{k-1} \).*

*Proof.* Consider a symbol \( x \) in a universal cycle \( U \). Note that there are \( \binom{n-1}{k-1} \) \( k \)-subsets of \( n \) that contain \( x \). Further, notice that \( x \) appears in \( k \) length \( k \) consecutive substrings of \( U \). Since these \( k \) consecutive substrings represent \( k \)-subsets of \( n \) that contain \( x \), we can conclude that \( k \) divides \( \binom{n-1}{k-1} \). \( \Box \)

In chapter two, we will build on Lemma 1.10 to create necessary divisibility conditions for a certain type of universal cycle.

Chung, Diaconis, and Graham concluded their work with the following open conjecture.

**Conjecture 1.11.** *Universal cycles for \( k \)-subsets of \( n \) always exist provided \( k \) divides \( \binom{n-1}{k-1} \) and \( n \geq n_0(k) \).*

It is a simple exercise to show that this is true for \( k = 2 \). Various authors have established existence of these universal cycles for small values of \( k \) such as 3, 4, and 6. However, limited progress has been made for general \( k \). Fan Chung, Persi Diaconis, and Ron Graham explain that we are “still completely baffled” by most values of \( k \). The techniques used to prove these known cases are interesting, but unfortunately do not seem to generalize to a theory that we can apply to all cases. Because we won’t use this material, the details are omitted. For more information consult [2].

### 1.4 Approach

Despite the difficulties that unordered sets present for constructing a transition graph, we will introduce an intermediate step that allows us to employ the traditional existence tools. To motivate the content of chapter two, we
present our approach in its entirety for the case of \( n = 8 \) and \( k = 3 \). Our goal is to create a transition graph approach for finding a universal cycle of 3-subsets of \( \{1, 2, 3, 4, 5, 6, 7, 8\} \).

Consider the set \( \Gamma \) below. This set has the interesting property that each size three subset of \( \{1, 2, 3, 4, 5, 6, 7, 8\} \) appears as a subset of exactly one element of \( \Gamma \). For instance, \( \{1, 2, 3\} \subset \{1, 2, 3, 4\} \) but \( \{1, 2, 3\} \not\subset \gamma \) for all \( \gamma \in \Gamma \setminus \{1, 2, 3, 4\} \).

<table>
<thead>
<tr>
<th>( \gamma \in \Gamma )</th>
<th>Label</th>
<th>( \gamma \in \Gamma )</th>
<th>Label</th>
</tr>
</thead>
<tbody>
<tr>
<td>{1, 2, 3, 4}</td>
<td>1234</td>
<td>{2, 3, 5, 8}</td>
<td>2358</td>
</tr>
<tr>
<td>{1, 2, 5, 6}</td>
<td>1256</td>
<td>{2, 3, 6, 7}</td>
<td>2376</td>
</tr>
<tr>
<td>{1, 2, 7, 8}</td>
<td>1287</td>
<td>{2, 4, 5, 7}</td>
<td>2754</td>
</tr>
<tr>
<td>{1, 3, 5, 7}</td>
<td>1357</td>
<td>{2, 4, 6, 8}</td>
<td>2486</td>
</tr>
<tr>
<td>{1, 3, 6, 8}</td>
<td>1368</td>
<td>{3, 4, 5, 6}</td>
<td>3456</td>
</tr>
<tr>
<td>{1, 4, 5, 8}</td>
<td>1458</td>
<td>{3, 4, 7, 8}</td>
<td>4837</td>
</tr>
<tr>
<td>{1, 4, 6, 7}</td>
<td>1467</td>
<td>{5, 6, 7, 8}</td>
<td>5678</td>
</tr>
</tbody>
</table>

Next we assign a labeling to the elements of \( \Gamma \). Given a set \( \{x_1, x_2, x_3, x_4\} \in \Gamma \), we arbitrarily assign a length 4 word, or label, containing the characters \( x_1, x_2, x_3, \) and \( x_4 \). For example, we can assign the word 1234 to \( \{1, 2, 3, 4\} \). The words 5678 and 4837 are valid labels for \( \{5, 6, 7, 8\} \) and \( \{3, 4, 7, 8\} \) respectively.

In the third step we consider each word label \( x_1x_2x_3x_4 \) and construct a corresponding de Bruijn cycle \((x_1x_2x_3x_4)\). That is we form a cycle \( C \) with vertex and edge sets

\[
V(C) = \{x_1x_2, \ x_2x_3, \ x_3x_4, \ x_4x_1\}
\]

\[
E(C) = \{x_1x_2x_3, \ x_2x_3x_4, \ x_3x_4x_1, \ x_4x_1x_2\}.
\]

The word 2547 forms the de Bruijn cycle (2547). This cycle contains the edges 725, 254, 547, and 472. The edge label 725 corresponds to the set \( \{2, 5, 7\} \). Furthermore, \( \{2, 5, 7\} \subset \{2, 4, 5, 7\} \).
Notice that the 3-subsets of \{2, 4, 5, 7\} are precisely the sets that correspond to the edges of (2547). This is true for any element \(\gamma \in \Gamma\). That is, given a cycle \(C\) constructed from a label of \(\gamma\), the edges of \(C\) correspond to the 3-subsets of \(\gamma\).

Let’s consider the graph \(T\) formed by the union of our cycles. Since \(\Gamma\) has the property that each size three subset of \{1, 2, 3, 4, 5, 6, 7, 8\} appears as a subset of exactly one element of \(\Gamma\), we know that every size three set corresponds to an edge in the graph \(T\).

<table>
<thead>
<tr>
<th>Cycles</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1234)</td>
</tr>
<tr>
<td>(1256)</td>
</tr>
<tr>
<td>(1287)</td>
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<td>(1357)</td>
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<td>(1368)</td>
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<td>(2486)</td>
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<td>(3456)</td>
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<tr>
<td>(4837)</td>
</tr>
<tr>
<td>(5678)</td>
</tr>
</tbody>
</table>

If this graph had an Eulerian circuit, then we could use its edge set to construct a universal cycle of 3-subsets of 8 integers. Unfortunately, the
graph $T$ is disconnected, which means it is not Eulerian. Thus we cannot form a universal cycle using $T$.

The cycle $(2754)$ is disconnected from the larger component of $T$. Notice that 24 is a vertex in $T$. If we relabel the set \{2, 4, 5, 7\} with the word 2475, then the cycle (2475) is connected to the larger component and the modified graph $T'$ is connected.

\[ T' \]

Since $T'$ satisfies the condition that $d^+(v) = d^-(v)$ for each vertex $v \in V(T')$ and has at most one nontrivial component, we know that $T'$ is Eulerian.

Indeed, we have the corresponding universal cycle

\[ 34563412561287123571467856713581368145823748376247524862. \]

In the following chapter we will formalize this construction to general $n$ and $k$ for which the approach is appropriate.
2 Universal Cycles for $k$-Subsets of $[n]

2.1 Cycle Sets

We begin by formalizing the idea of a cycle set. We will employ bracketed $n$ to denote the integers 1 through $n$, i.e. $[n] = \{1, \ldots, n\}$. We use $\binom{n}{k}$ to denote all $k$-subsets of $[n]$. Note that $|\binom{n}{k}| = \binom{n}{k}$.

**Definition 2.1.** Given $n$ and $k$ positive integers such that $n > k$, a *dimension $k$ cycle set of $n$* is a set $^k\Gamma_n \subseteq \binom{n}{k+1}$ such that for every $\delta \in \binom{n}{k}$ there exists a unique $\gamma \in ^k\Gamma_n$ with $\delta \subset \gamma$. If $n$ and $k$ are not specified, we may write a generic cycle set as $\Gamma$.

**Example 2.2.** Let $n = 8$ and $k = 3$. The list in Figure 2.3 provides a valid dimension 3 cycle set of 8.

**Figure 2.3. $^3\Gamma_8$**

\[
\begin{align*}
\{1, 2, 3, 4\} & \quad \{1, 2, 5, 6\} \\
\{1, 2, 7, 8\} & \quad \{1, 3, 5, 8\} \\
\{1, 3, 6, 7\} & \quad \{1, 4, 5, 7\} \\
\{1, 4, 6, 8\} & \quad \{2, 3, 5, 7\} \\
\{2, 3, 6, 8\} & \quad \{2, 4, 5, 8\} \\
\{2, 4, 6, 7\} & \quad \{3, 4, 5, 6\} \\
\{3, 4, 7, 8\} & \quad \{5, 6, 7, 8\}
\end{align*}
\]

Notice that each set in $[\binom{8}{3}]$ appears exactly once as a subset of an element of $^3\Gamma_8$. For example $\{1, 6, 8\} \in \binom{8}{3}$ appears as a subset of $\{1, 4, 6, 8\}$ but is not contained in any other element of $^3\Gamma_8$.

Notice that there are four proper subsets of size three in an element $\gamma \in ^3\Gamma_8$. For example, the sets $\{2, 4, 6\}$, $\{2, 4, 7\}$, $\{2, 6, 7\}$, and $\{4, 6, 7\}$ are all the 3-subsets of $\gamma = \{2, 4, 6, 7\}$.

This observation motivates a basic fact about all cycle sets.

**Lemma 2.4.**

\[
|\ ^k\Gamma_n | = \frac{\binom{n}{k}}{\binom{k+1}{k}} = \frac{\binom{n}{k}}{k+1}.
\]
Each element $\gamma$ of a cycle set has size $k + 1$. There are $\binom{k+1}{k}$ $k$-subsets contained in $\gamma$. Since each element of $[n]^k$ appears exactly once as a subset in $^k\Gamma_n$, the size of $^k\Gamma_n$ is the size of $\binom{n}{k}$ divided $k + 1$ ways.

**Example 2.5.** By Lemma 2.4, we know that size of a cycle set $^3\Gamma_8$ is 14. Further, we know that $|2\Gamma_7| = 7$, $|3\Gamma_{10}| = 30$, and $|4\Gamma_{11}| = 66$. Note that the cardinality of a cycle set quickly increases with a slight change in $n$ or $k$. For instance, $|3\Gamma_{10}| - |3\Gamma_8| = 16$.

Continuing with our example of a dimension 3 cycle set of 8 from Figure 2.3, we can inspect the size of the intersection between any two elements.

Figure 2.6 provides the intersection size of $\gamma_1 = \{1, 3, 6, 7\}$ with each set in $^3\Gamma_8$.

**Figure 2.6.** $\gamma_1$'s intersection size with elements of $^3\Gamma_8$.

| $\gamma_i$ | $|\gamma_1 \cap \gamma_i|$ |
|------------|------------------|
| $\gamma_i = \{1, 3, 6, 7\}$ | four |
| $\gamma_i \in ^3\Gamma_8 \setminus \{\{1, 3, 6, 7\}, \{2, 4, 5, 8\}\}$ | two |
| $\gamma_i = \{2, 4, 5, 8\}$ | 0 |

Notice that no elements in $^3\Gamma_8$ have size three intersection with $\gamma_1$. The definition of a cycle set allows us to generalize this observation.

**Lemma 2.7.** If $\gamma_i, \gamma_j \in ^k\Gamma_n$, then $|\gamma_i \cap \gamma_j| \leq k - 1$ for $i \neq j$.

**Proof.** Recall that the definition of a cycle set states that a size $k$ subset of $[n]$ occurs exactly once as a subset of an element of cycle set $\Gamma$. Let $\gamma_i$, and $\gamma_j$ be distinct elements of $\Gamma$. If $|\gamma_i \cap \gamma_j| = k + 1$, then $\gamma_i = \gamma_j$. If $|\gamma_i \cap \gamma_j| = k$, then $\gamma_i \neq \gamma_j$, but $\gamma_i$, and $\gamma_j$ share a size $k$ subset in common. This contradicts the definition of a cycle set. Thus $|\gamma_i \cap \gamma_j| \leq k - 1$.

We have made some nice observations using the elements in 2.3. We can also consider the elements that are missing from $^3\Gamma_8$.

Let’s say our favorite size four subset of $[8]$ is $\{1, 5, 6, 7\}$. The cycle set given in Figure 2.3, let’s call it $\Gamma_1$, doesn’t include this element, but we want a cycle set that does. We can define a function $f : [n] \rightarrow [n]$ such that $f(\Gamma_1) = \Gamma_2$, a cycle set that does contain the element $\{1, 5, 6, 7\}$. Let $f$ map $1 \rightarrow 1, 2 \rightarrow 4, 3 \rightarrow 4, 4 \rightarrow 1, 5 \rightarrow 5, 6 \rightarrow 4, 7 \rightarrow 4, 8 \rightarrow 2$. 

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Our function \( f \) maps the element \( \{1, 2, 3, 4\} \) to \( \{7, 5, 6, 1\} = \{1, 5, 6, 7\} \) (since sets are unordered). Figure 2.8 shows the mapping of \( \Gamma_1 \) onto a valid cycle set, namely \( \Gamma_2 \).

**Figure 2.8.** \( \Gamma_1 \) maps onto \( \Gamma_2 \).

\[
\begin{array}{c|c}
\Gamma_1 & \Gamma_2 \\
\hline
\{1, 2, 3, 4\} & \{1, 5, 6, 7\} \\
\{1, 2, 5, 6\} & \{3, 5, 7, 8\} \\
\{1, 2, 7, 8\} & \{2, 4, 5, 7\} \\
\{1, 3, 5, 8\} & \{2, 3, 6, 7\} \\
\{1, 3, 6, 7\} & \{4, 6, 7, 8\} \\
\{1, 4, 5, 7\} & f \rightarrow \{1, 3, 4, 7\} \\
\{1, 4, 6, 8\} & \rightarrow \{1, 2, 7, 8\} \\
\{2, 3, 5, 7\} & \{3, 4, 5, 6\} \\
\{2, 3, 6, 8\} & \{2, 5, 6, 8\} \\
\{2, 4, 5, 8\} & \{1, 2, 3, 5\} \\
\{2, 4, 6, 7\} & \{1, 4, 5, 8\} \\
\{3, 4, 5, 6\} & \{1, 3, 6, 8\} \\
\{3, 4, 7, 8\} & \{1, 2, 4, 6\} \\
\{5, 6, 7, 8\} & \{2, 3, 4, 8\} \\
\end{array}
\]

Again, our example motivates a fact about all cycle sets.

**Lemma 2.9.** If a dimension \( k \) cycle set of \( n \) exists, then for any arbitrary size \( k+1 \) set \( \omega \subseteq [n] \) there exists some cycle set \( \Gamma \) such that \( \omega \) is an element of \( \Gamma \).

Before we prove this lemma, we need a proposition.

**Proposition 2.10.** Let \( \Omega \in \mathcal{P}(\binom{[n]}{k+1}) \) such that \( |\Omega| = \binom{n}{k} / (k+1) \). There exists some \( \delta_m \in \binom{[n]}{k} \) such that there exists no \( \omega \in \Omega \) with \( \delta_m \subset \omega \) if and only if there exists some \( \delta_d \in \binom{[n]}{k} \) such that there are \( \omega_1, \omega_2 \in \Omega \) with \( \delta_d \subset \omega_1 \) and \( \delta_d \subset \omega_2 \).

**Proof.** Consider a set \( \Omega \). Each \( \omega \in \Omega \) contains \( k+1 \) distinct \( k \)-subsets. Since \( |\Omega| = \binom{n}{k} / (k+1) \) we can conclude that the elements of \( \Omega \) have \( \binom{n}{k} \) \( k \)-subsets \( \delta_i \) (allowing for repetition). Let’s say there exists some \( \delta_m \in \binom{[n]}{k} \) such that there exists no \( \omega \in \Omega \) with \( \delta_m \subset \omega \). Recall that \( |\binom{[k]}{k}| = \binom{n}{k} \). Since \( \delta_m \) is missing as a subset of an \( \omega \), and there are \( \binom{n}{k} \) \( k \)-subsets covered by \( \Omega \), there must be some \( \delta_d \) such that for \( \omega_1, \omega_2 \in \Omega \) we have \( \delta_d \subset \omega_1 \) and \( \delta_d \subset \omega_2 \).
Similarly, let’s say that there exists some \( \delta_d \in \binom{n}{k} \) such that there are \( \omega_1, \omega_2 \in \Omega \) with \( \delta_d \subset \omega_1 \) and \( \delta_d \subset \omega_2 \). Since \( |\Omega| = \binom{n}{k}/(k+1) \) we can conclude that the elements of \( \Omega \) have \( \binom{n}{k} \) \( k \)-subsets \( \delta_i \) (allowing for repetition). Since \( |\binom{n}{k}| = \binom{n}{k} \) and \( \delta_d \) occurs twice as a subset, then there must be some \( \delta_m \) missing as a subset of an \( \omega \). \( \square \)

Now we can prove Lemma 2.9.

**Proof.** Let \( \Gamma \) be a dimension \( k \) cycle set of \( n \) and let \( \{z_1, z_2, \ldots, z_{k+1}\} \) be an arbitrary size \( k+1 \) subset of \([n]\). Choose an arbitrary element \( \gamma = \{a_1, a_2, \ldots, a_{k+1}\} \) from \( \Gamma \). Let function \( f \) be a permutation map (i.e., a map \([n] \to [n]\) for which every element of \([n]\) occurs exactly once as an image value). Thus \( f \) is a bijection from \([n]\) onto itself. We can choose \( f \) such that \( a_i \mapsto z_i \) for all \( 1 \leq i \leq k+1 \). Therefore \( \{z_1, z_2, \ldots, z_{k+1}\} \) occurs as an element in \( f(\Gamma) \).

All that remains to be verified is that \( f(\Gamma) \) is in fact a cycle set. Assume for a contradiction that \( f(\Gamma) \) is not a cycle set. So there exists a \( \delta \in \binom{n}{k} \) without a unique \( f(\gamma) \in f(\Gamma) \) for which it is a subset. By Proposition 2.10, there exists some \( \delta_d \in \binom{n}{k} \) that appears as a subset of distinct \( f(\gamma_1), f(\gamma_2) \in f(\Gamma) \).

Let one copy of \( \delta_d \) occur in \( f(\gamma_1) = \delta_d \cup \{b_1\} \in f(\Gamma) \) and the other in \( f(\gamma_2) = \delta_d \cup \{b_2\} \in f(\Gamma) \) for \( b_1 \neq b_2 \). Since \( f \) is an injective function, we can conclude that \( f(\gamma_1) \neq f(\gamma_2) \) means \( \gamma_1 \neq \gamma_2 \). Consider \( f^{-1}(\delta_d) \). Since \( f \) is injective, there exists exactly one \( f^{-1}(\delta_d) \in \binom{n}{k} \) such that \( f(f^{-1}(\delta_d)) = \delta_d \). Therefore \( f^{-1}(\delta_d) \subset \gamma_1 \) and \( f^{-1}(\delta_d) \subset \gamma_2 \). Thus \( |\gamma_1 \cap \gamma_2| = k \). This is a contradiction. \( \square \)

**Corollary 2.11.** If a dimension \( k \) cycle set of \( n \) exists, then \( ^k\Gamma_n \) is not unique.

**Proof.** We know that

\[
|\binom{n}{k+1}| = \binom{n}{k+1} = \frac{n!}{(n-k-1)!(k+1)!}.
\]

By Lemma 2.4 we have

\[
|^k\Gamma_n| = \frac{\binom{n}{k}}{k+1} = \frac{n!}{(n-k)!(k+1)!}.
\]

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Thus, for all \( k \geq 1, \)
\[
\left| \left[ \frac{n}{k+1} \right] \right| > |k\Gamma_n|.
\]

Therefore, given any \( k > 0, \) there exists some size \( k+1 \) set not contained in \( \Gamma. \) Given a size \( k+1 \) set not contained in \( \Gamma, \) the desired result follows from Lemma 2.9. If one set \( \Gamma_1 \) does not contain an element from \( \left[ \frac{n}{k+1} \right] \) then 2.9 tells us there exists another set \( \Gamma_2 \) that does. The two cycles sets are distinct. \( \square \)

We have examined an example of \( 3\Gamma_8. \) Does a cycle set always exist for any \( n \) and \( k? \) With very little effort we can find a pair of \( n \) and \( k \) such that no dimension \( k \) cycle set of \( n \) exists. Examples 2.12 and 2.13 give us two scenarios where \( n \) and \( k \) have no cycle set.

**Example 2.12.** Consider \( n = 6 \) and \( k = 2. \) Without loss of generality we’ll examine the sets from \( \left[ \frac{6}{2} \right] \) containing \( \{1\}. \) There are five of them, namely \( \{1,2\}, \{1,3\}, \{1,4\}, \{1,5\}, \) and \( \{1,6\}. \) If a cycle set existed, then each of these size two sets would appear as a subset of a size three set. Notice that if \( \{1,2\} \subset \{1,2,a\} \) for some \( a \in [6], \) then it is also true that \( \{1,a\} \subset \{1,2,a\}. \) Thus size two sets containing \( \{1\} \) appear two at a time in elements of \( \Gamma. \) Since 5 is not divisible by 2, there is no \( \Gamma \) set.

This example suggests a divisibility condition for \( n \) and \( k. \)

**Example 2.13.** Consider \( n = 6 \) and \( k = 3. \) Without loss of generality we’ll look at the sets in \( \left[ \frac{6}{3} \right] \) that contain \( \{1,2\}, \) i.e., the sets \( \{1,2,3\}, \{1,2,4\}, \{1,2,5\}, \) and \( \{1,2,6\}. \) We can make an observation similar to the one we made in Example 2.11. Notice if \( \{1,2,3\} \subset \{1,2,3,a\} \) for \( a \in [6], \) then \( \{1,2,a\} \subset \{1,2,3,a\}. \) We can conclude that if one size three set containing \( \{1,2\} \) appears in an element \( \gamma, \) then there is a second size three set containing \( \{1,2\} \) in \( \gamma. \)

Let’s assume a cycle set \( \Gamma \) does exist. Without loss of generality let \( \{1,2,3,4\} \in \Gamma. \) Thus \( \{1,2,5\} \) and \( \{1,2,6\} \) are subsets of the same element \( \gamma \in \Gamma. \) Thus \( \{1,2,5,6\} \) must be an element of \( \Gamma. \)

Now consider the size three sets containing \( \{1,3\} \) that we know must be subsets of \( \gamma \) elements. The sets \( \{1,2,3\} \) and \( \{1,3,4\} \) appear as subsets of \( \{1,2,3,4\}. \) The remaining size three sets containing \( \{1,3\} \) are \( \{1,3,5\}, \) and \( \{1,3,6\}. \) Since this type of three set appears two at a time in elements of \( \Gamma \) we can conclude that \( \{1,3,5,6\} \in \Gamma. \) Since \( \{1,5,6\} \subset \{1,2,5,6\}, \) that would mean \( \{1,5,6\} \) appears twice as a subset in \( \Gamma. \) Thus no dimension 3 cycle set of 6 exists.
The values of \( n \) and \( k \) from Example 2.12 and 2.13 fail to support a cycle set for different reasons. The first example is clearly a divisibility problem. In the second, the size-3 sets containing certain two element subsets must be divided among size-4 sets. The problem here is that in order to add all sets from \([6\choose3]\), repetition of certain size-3 sets becomes necessary. Despite this seeming difference, both Example 2.12 and Example 2.13 can be proven invalid using the same divisibility condition.

**Lemma 2.14.** If \( k\Gamma_n \exists \) exists for \( n \) and \( k \), then for all \( m \in \{0, 1, \ldots, k - 1\} \),

\[
(k + 1 - m) \left| \binom{n - m}{k - m} \right.
\]

*Proof.* We claim that when the set \( \{z_1, \ldots, z_m\} \) is a subset of \( \gamma \in k\Gamma_n \) it is contained in \( \binom{k+1-m}{k-m} \) \( k \)-subsets of \( \gamma \). To see this, if \( \gamma \) contains at least \( m \) elements, fix \( m \) of them. Then choose \( k - m \) elements from the remaining \( k + 1 - m \) elements in \( \gamma \) to fill out a size \( k \) set.

We claim that the integers \( \{z_1, \ldots, z_m\} \) are contained in \( \binom{n-m}{k-m} \) \( k \)-subsets of \( [n] \). To show this, again fix \( m \) elements. Then choose \( k - m \) elements from the remaining \( n - m \) elements in \( [n] \) to fill out a size \( k \) set.

The elements from \( [n] \) that contain \( \{z_1, \ldots, z_m\} \) appear as subsets of \( \gamma \) elements \( k + 1 - m \) at a time. Thus \( (k + 1 - m) \) divides \( \binom{n-m}{k-m} \).

\( \square \)

**Example 2.15.** Let’s return to Example 2.12 and 2.13 and use Lemma 2.14 to show that a cycle set does not exist for \( n \) and \( k \). For \( n = 6, k = 2, \) and \( m = 1 \), we already noticed that \( 2 \nmid \binom{5}{1} = 5 \). For \( n = 6, k = 3, \) and \( m = 1 \), we have \( 3 \nmid \binom{5}{2} = 10 \). We can conclude that no cycle set exists for \( n = 6 \) with \( k = 2, 3 \).

### 2.2 Cycle Set Existence

#### 2.2.1 Hypergraphs

The condition given in Lemma 2.14 is necessary, but it is unclear if it is sufficient for the existence of a dimension \( k \) cycle set of \( n \). To better understand cycle sets we can establish a one-to-one correspondence with a well-known class of objects in graph theory: hypergraphs. Hypergraphs are a generalization of a graph because we allow an edge to connect any number of vertices.
Example 2.16. Consider the hypergraph $H$ below. This example has vertex set $V(H) = \{v_i : 1 \leq i \leq 7\}$ and edge set $E(H) = \{e_j : 1 \leq j \leq 5\}$.

We say vertices $v_2$ and $v_3$ are contained in $e_1$. We write $\{v_1, v_3\} \subset e_1$. The degree $d(v_i)$ of vertex $v_i$ is the number of edges that contain it. For example, $d(v_1) = 3$. The size of an edge is the number of vertices it contains. So edge $e_2$ has size 5.

Definition 2.17. Given a hypergraph $H$, a transversal $\tau$ is a set of vertices that covers $E(H)$. In other words, given any $e \in E(H)$ there is a $v \in \tau$ such that $v \subset e$.

Example 2.18. One transversal in $H$ from Example 2.16 is $\tau = \{v_3, v_5, v_7\}$. Vertex $v_3$ covers $e_1$, $e_2$, and $e_4$. Vertex $v_5$ covers $e_2$ and $e_5$. Vertex $v_7$ covers $e_2$ and $e_3$. Thus each edge in $H$ is covered and $\tau$ is a transversal. Notice that edge $e_2$ is not covered uniquely. In fact, each vertex $v \in \tau$ covers $e_2$.

Definition 2.19. Let $H([k]_n, [\frac{n}{k+1}])$ be a hypergraph with vertex set $V = [k+1]_n$ and edge set $E = [\frac{n}{k}]$. Then each vertex $v \in V$ is a unique element $\omega \in [k+1]_n$ and each edge in $E$ is a unique element $\delta \in [\frac{n}{k}]$. Let $v \subset e$ if and only if $\delta \subset \omega$.

Note that this definition is inclusion-reversing. We are saying that vertex $v$ is contained in edge $e$ when the set $e$ is contained in set $v$.

Example 2.20. Consider the vertex $v_1$ from hypergraph $H([\frac{4}{4}], [\frac{4}{3}])$. 

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We can use the edges \( \{1, 2, 3\} \) and \( \{2, 3, 4\} \) to deduce the set \( v_1 \). Notice that \( \{1, 2, 3\} \subseteq v_1 \) and \( \{2, 3, 4\} \subseteq v_1 \). Thus \( v_1 = \{1, 2, 3, 4\} \).

**Example 2.21.** Consider the edge \( \{1, 2\} \) from hypergraph \( H([n], [k]) \).

![Diagram](https://via.placeholder.com/150)

Consider the vertices contained in edge \( \{1, 2\} \). Each vertex is contained in three edges. For example, vertex \( \{1, 2, 6\} \) is contained in edges \( \{1, 2\} \), \( \{1, 6\} \), and \( \{2, 6\} \). Thus the degree of vertex \( \{1, 2, 6\} \) is three. We can use this observation to formulate a fact about all hypergraphs \( H([n], [k]) \) for all \( n \) and \( k \).

**Definition 2.22.** We say a hypergraph \( H \) is \( j \)-regular if every vertex in \( V(H) \) has degree \( j \).

**Lemma 2.23.** The hypergraph \( H([n], [k]) \) is \( k + 1 \) regular.

*Proof.\* Consider \( v \in [n]_{k+1} \). There are \( \binom{k+1}{k} \) distinct \( k \)-subsets of \( v \). By definition of \( H([n], [k]) \), the vertex \( v \) is contained in the edges that are those size \( k \) sets. Thus \( v \) is contained in \( k + 1 \) edges. \( \square \)
Example 2.24. Continuing with hypergraph from Example 2.21, consider the size of edge \{1, 2\}. There are 4 vertices in this edge because \{1, 2\} is contained in 4 elements of \([\frac{n}{3}]\). This is true of every edge in \(H\). Further, we can make a statement about all hypergraphs \(H([\frac{n}{k+1}], [\frac{n}{k}])\) for all \(n\) and \(k\).

Definition 2.25. We say a hypergraph \(H\) is \(l\)-uniform if all edges have size \(l\).

Lemma 2.26. The hypergraph \(H([\frac{n}{k+1}], [\frac{n}{k}])\) is \(n-k\) uniform.

Proof. Consider edge \(e \in [\frac{n}{k}]\). There are \(n-k\) integers not in \(e\). Thus there are \(n-k\) size \(k+1\) sets that contain \(e\). Thus there are \(n-k\) vertices that are contained in edge \(e\). \(\square\)

We can relate a cycle set to a particular transversal of \(H\).

Lemma 2.27. A dimension \(k\) cycle set of \(n\) exists if and only if \(H([\frac{n}{k+1}], [\frac{n}{k}])\) has a transversal \(\tau\) such that

\[|\tau| = \frac{\binom{n}{k}}{k+1}.
\]

Proof. (\(\Rightarrow\)) Assume a cycle set \(\Gamma\) exists. Then \(\Gamma\) is a set containing \(k+1\)-subsets of \([n]\) such that for every set \(\delta \in [\frac{n}{k}]\) there is a unique \(\gamma \in \Gamma\) such that \(\delta \subset \gamma\). Let \(\tau = \Gamma\). For all \(e \in E(H)\) there is a \(v \in \tau\) such that \(e \subset v\) because edge \(e\) corresponds to \(\delta\), vertex \(v\) corresponds to \(\gamma\), and \(\delta \subset \gamma\). Note also \(|\tau| = |k\Gamma_n| = \binom{n}{k}/(k+1)\).

(\(\Leftarrow\)) Assume \(H\) has a transversal \(\tau\) of size \(\binom{n}{k}/(k+1)\). Notice that \(\deg(v) = k+1\) for all \(v \in V(H)\) since \(v\) is a size \(k+1\) set and this set has \(k+1\) distinct \(k\)-subsets. Then there are \(k+1\) edges \(e\) such that \(v \subset e\). Thus \(\tau\) covers \(\binom{n}{k}\) edges (allowing for repetition). Since \(|E(H)| = \binom{n}{k}\) and \(\tau\) is a transversal, each edge must be covered exactly once. Thus for all \(e \in E(H)\) there is a unique \(v \in \tau\) such that \(e \subset v\). Therefore \(\tau\) is a cycle set. \(\square\)

2.2.2 The Inclusion-Exclusion Principle

For general values of \(n\) and \(k\) it is difficult to determine if a valid cycle set exists, let alone the exact number of distinct cycle sets. However, \(k = 1\) provides a unique case.
We know by Lemma 2.14 that a cycle set for the parameter $k = 1$ must have $n \in 2\mathbb{N}$ so that $(k+1)\binom{n}{k}$. We can also verify that a dimension 1 cycle set exists for every $n \in 2\mathbb{N}$. To construct a valid set $\Gamma_n$, we remove integers $a$ and $b$ two at a time from $[n]$ and form elements $\gamma = \{a, b\}$. Any such set is a cycle set. Furthermore, any cycle set of $k = 1$ for $n$ even must be of this form. As a consequence, the parameter $k = 1$ has the property that we have a formula that counts dimension 1 cycle sets of $n$. One valid cycle set is

$$\Gamma = \{\{1, 2\}, \{3, 4\}, \ldots, \{n - 1, n\}\}.$$

**Proposition 2.28.** For $k = 1$ and $n$ even, the number of distinct cycle sets is

$$\frac{\binom{n}{2, 2, \ldots, 2}}{\binom{n}{k+1}!}.$$

For values of $k > 1$, however, this greedy approach generally fails to generate a cycle set, and we have no known formula for the number of cycle sets of dimension $k$. Consider the following greedy algorithm: Construct a set $H$ by selecting any element $\gamma_1 \in [\frac{n}{k+1}]$ and adding $\gamma_1$ to $H$. Continue by selecting an arbitrary $\gamma_j \in [\frac{n}{k+1}]$. If $\gamma_i \cap \gamma_j \leq k - 1$ for all $\gamma_i \in H$, then add $\gamma_j$ to $H$. If $\gamma_i \cap \gamma_j = k$ for some $\gamma_i \in H$, then choose a different $\gamma_j \in [\frac{n}{k+1}]$. We continue in this fashion until we can add no more elements to $H$.

For $n = 8$ and $k = 3$, we can have the following set $H$ after the $10^{th}$ iteration of the algorithm.

$$\{1, 2, 4, 8\} \quad \{2, 3, 4, 6\}$$
$$\{1, 2, 5, 7\} \quad \{2, 3, 7, 8\}$$
$$\{1, 3, 4, 5\} \quad \{2, 5, 6, 8\}$$
$$\{1, 3, 6, 8\} \quad \{3, 5, 6, 7\}$$
$$\{1, 4, 6, 7\} \quad \{4, 5, 7, 8\}$$

Although $|H| \neq \binom{n}{k}/(k+1)$, we can add no more elements from $[\frac{n}{k+1}]$ to $H$ such that the condition $\gamma_i \cap \gamma_j \leq k - 1$ is satisfied. Thus the greedy algorithm has failed to generate a cycle set.

For fixed values of $n$ and $k$, the principle of inclusion-exclusion can be used to prove the existence of a cycle set. However, the process is labor-intensive and difficult to represent in a concise formula. Throughout this section we will provide examples using $n = 8$, $k = 3$ to illustrate how one
might use the inclusion-exclusion process to prove the existence of a cycle set.

Of course we already know that a set $^3\Gamma_8$ exists because we provided one in Figure 2.3. On the other hand, the parameters 3 and 8 are small enough that illustrating the concept is manageable, yet large enough that finding a cycle set is nontrivial.

The following formulation will direct our use of the inclusion-exclusion principle. (More general information about inclusion-exclusion can be found in Appendix A.) The idea is to count all possible $\binom{n}{k}\bigg/(k+1)$ size sets containing elements of $\left[\frac{n}{k+1}\right]$ and then to count the $\binom{n}{k}\bigg/(k+1)$ size sets that don’t satisfy the conditions of a cycle set. By taking the difference we know how many distinct cycle sets exist for $n$ and $k$.

**Definition 2.29.** Let

$$S = \left\{ \Omega \in \mathcal{P}\left(\left[\frac{n}{k+1}\right]\right) : |\Omega| = \frac{\binom{n}{k}}{k+1} \right\}.$$  

One can easily see the following.

**Lemma 2.30.**

$$|S| = \binom{n}{k+1}\bigg/\binom{n}{k+1}$$.

**Example 2.31.** Consider $n = 8$ and $k = 3$. Then

$$|S| = \binom{70}{14} = 193, 253, 756, 909, 160.$$  

If we define a collection of sets $\{A_\alpha\}$ such that $\bigcup A_\alpha$ contains exactly the sets $\Omega$ in $S$ that do not satisfy the conditions of a cycle set, then a cycle set exists in $S$ if and only if $|S - (\bigcup A_\alpha)| > 0$.

**Definition 2.32.** Let $\omega_i, \omega_j$ be distinct elements of $\left[\frac{n}{k+1}\right]$. If $|\omega_i \cap \omega_j| = k$, then $A_{\omega_i, \omega_j} = \{\Omega \in S : \omega_i \in \Omega$ and $\omega_j \in \Omega\}$. If $|\omega_i \cap \omega_j| < k$, then $A_{i,j} = \emptyset$.

In other words, $A_{\omega_i, \omega_j}$ counts the sets $\Omega \in S$ that fail to be a cycle set as a direct result of the elements $\omega_i$ and $\omega_j$ appearing in $\Omega$. We often abbreviate $A_{\omega_i, \omega_j}$ as $A_{i,j}$. 
Lemma 2.33. The set $\bigcup_{i \neq j} A_{i,j}$ contains exactly the sets $\Omega$ in $S$ that do not satisfy the conditions of a cycle set.

Proof. Assume for a contradiction that there exists some $\Omega \not\in \bigcup A_{i,j}$ that does not satisfy the conditions of a cycle set. Then there exists some $\delta_m \in [\frac{n}{k}]$ such that there exists no $\omega \in \Omega$ with $\delta_m \subset \omega$. or there is some $\delta_d \in [\frac{n}{k}]$ such that there are $\omega_1, \omega_2 \in \Omega$ with $\delta_d \subset \omega_1$ and $\delta_d \subset \omega_2$. In the latter case $\Omega$ would be contained in $A_{1,2}$. So assume $\Omega$ is missing some $\delta_d \in [\frac{n}{k}]$. By Proposition 2.10 we know that $\delta_m$ missing implies there exists some $\delta_d \in [\frac{n}{k}]$ such that there are $\omega_1, \omega_2 \in \Omega$ with $\delta_d \subset \omega_1$ and $\delta_d \subset \omega_2$. Thus $\Omega$ is contained in $A_{1,2}$. This is a contradiction.

Assume for a contradiction that $\Omega$ is a cycle set such that $\Omega \in \bigcup A_{i,j}$. Thus $\Omega \in A_{1,2}$ for some $\omega_1, \omega_2 \in [\frac{n}{k+1}]$. That means there is some $\delta_d \in [\frac{n}{k}]$ such that for $\omega_1, \omega_2 \in \Omega$ we have $\delta_d \subset \omega_1$ and $\delta_d \subset \omega_2$. This is a contradiction. □

Example 2.34. Let $n = 8$ and $k = 3$. For $\omega_1 = \{1,2,3,8\}$ and $\omega_2 = \{1,2,3,7\}$, let’s look at an example of sets $\Omega_a, \Omega_b \in A_{\omega_1, \omega_2}$.

<table>
<thead>
<tr>
<th>$\Omega_a$</th>
<th>$\Omega_b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>${1,2,3,4}$</td>
<td>${1,2,3,6}$</td>
</tr>
<tr>
<td>${1,2,3,8}$</td>
<td>${2,3,6,8}$</td>
</tr>
<tr>
<td>${1,2,3,7}$</td>
<td>${2,4,5,6}$</td>
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<td>${1,3,5,7}$</td>
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</tr>
<tr>
<td>${1,4,5,8}$</td>
<td>${4,6,7,8}$</td>
</tr>
<tr>
<td>${1,4,6,7}$</td>
<td>${5,6,7,8}$</td>
</tr>
</tbody>
</table>

Notice that $\Omega_a \in A_{1,2}$ because $\{1,2,3,7\}, \{1,2,3,8\} \in \Omega_a$ and $| \{1,2,3,7\} \cap \{1,2,3,8\} | = 3$.

Similarly, $\Omega_b \in A_{1,2}$ because $\{1,2,3,7\}, \{1,2,3,8\} \in \Omega_b$.

Now consider $A_{1,3}$ for $\omega_3 = \{2,3,4,8\}$. Since $\{1,2,3,8\}, \{2,3,4,8\} \in \Omega_b$ and $| \{1,2,3,8\} \cap \{2,3,4,8\} | = 3$, we know that $\Omega_b \in A_{1,3}$. Since $A_{1,2} \cap A_{1,3} \neq \emptyset$, we must count the intersection to avoid overcounting the sets that do not satisfy the conditions of a cycle set.

We’ve seen that $\Omega_b$ is not a cycle set because there are multiple size $k$ sets that appear as a subset of more than one element $\omega \in \Omega_b$. Further, the set $\Omega_b$ is not a cycle set because it has no element $\omega$ such that $\{1,2,4\} \subset \omega$.

We have one last important observation to make. Notice that there is no element $\omega \in \Omega_a$ such that $\{2,3,7\} \subset \omega$. Thus one of the ways that $\Omega_a$
does not qualify as a cycle set is because it is missing \( \{2, 3, 7\} \). If \( \Omega_a \) fails to be a cycle set because a certain size \( k \) set is missing, one may ask where we account for this particular set in \( \cup A_{i,j} \). Notice that \( \Omega_a \) was counted in \( |A_{1,2}| \). This should not be surprising by Lemma 2.33.

**Corollary 2.35.** Given \( n \) and \( k \), if \( \binom{n}{k+1} = 1 \), then the number of distinct cycle sets is \( |S| \).

**Proof.** Since \( \binom{n}{k}/(k+1) = 1 \), there is no \( \binom{n}{k}/(k+1) \) sized set \( \Omega \) that contains two elements. Thus \( A_{i,j} = \emptyset \) for all \( i, j \). Thus by the principle of inclusion-exclusion, \( |S - (\cup A_{i,j})| = |S| \).

**Example 2.36.** We can verify Proposition 2.28, Corollary 2.35 and Lemma 2.30 by considering \( n = 2 \), and \( k = 1 \). We know that there is exactly one cycle set, namely \( \{\{1,2\}\} \). Since there is only one element in \( [\frac{n}{2}] \), we know \( A_{i,j} = \emptyset \) for all \( i, j \). Thus for \( n = 2 \) and \( k = 1 \), we must have

\[
\binom{2,2,\ldots,2}{k+1} = \binom{n}{k+1}
\]

Indeed,

\[
\frac{2!}{1!} = \binom{2}{1} = 1.
\]

Based on the definition of \( S \) and \( A_{i,j} \), we will employ a generalized version of inclusion-exclusion where we take a finite set \( S \) and a collection of subsets \( A_{i} \) and we compute \( |S - (\cup A_{i})| \). The relevant equation to keep in mind states that given \( A_{1}, A_{2}, \ldots, A_{m} \subseteq S \) where \( S \) is a finite set, we have

\[
|S - (A_{1} \cup A_{2} \cup \cdots \cup A_{m})| = \sum_{I \subseteq [m]} (-1)^{|I|}|A_{I}|	ext{ where }A_{I} = \cap_{i \in I} A_{i}.
\]

For this calculation it is useful to make a few observation about the sets in \([\binom{n}{k+1}]\). Consider \( \delta = \{1, 2, 3\} \in [\frac{n}{3}] \). We know that \( \delta \) is a subset of \( n - k = 5 \) elements of \([\frac{n}{3}]\). Those sets are \( \{1, 2, 3, 4\}, \{1, 2, 3, 5\}, \{1, 2, 3, 6\}, \{1, 2, 3, 7\}, \text{ and } \{1, 2, 3, 8\} \). This follows from Lemma 2.26, which states that given \( \delta \in [\frac{n}{k}] \), there are \( n - k \) sets \( \omega \in [\frac{n}{k+1}] \) such that \( \delta \subseteq \omega \).

We need one more useful fact before we proceed with the inclusion-exclusion principle.
**Proposition 2.37.** If $\delta_1$ and $\delta_2 \in [k]^{n}$ are distinct subsets of $\omega_i \in [k+1]^n$ then there is no $\omega_j \in [k+1]^n \setminus \{\omega_i\}$ such that $\delta_1 \subset \omega_j$ and $\delta_2 \subset \omega_j$.

**Proof.** To see this consider the fact that $\delta_1$ and $\delta_2$ are distinct. Thus they share at most $k - 1$ elements in common. Thus $|\delta_1 \cup \delta_2| \geq k + 1$. Since $\delta_1 \cup \delta_2$ is contained inside the size $k + 1$ set $\omega_i$, then $|\delta_1 \cup \delta_2| \leq k + 1$. Thus $|\delta_1 \cup \delta_2| = k + 1$.

If $\delta_1$ and $\delta_2$ both appear in $\omega_i$ and in $\omega_j$, then $\delta_1 \cup \delta_2 \subset \omega_i$ and $\delta_1 \cup \delta_2 \subset \omega_j$. Since $|\delta_1 \cup \delta_2| = k + 1$, we can conclude that $\omega_i = \omega_j$. 

Now we have the relevant information to begin evaluating $|S - (\bigcup_{i \neq j} A_{i,j})|$, i.e., the formula

$$|S| - \sum |A_{i,j}| + \sum |A_{i,j} \cap A_{l,m}| - \sum |A_{i,j} \cap A_{l,m} \cap A_{o,p}| + \cdots + (-1)^{\binom{n+1}{2}} \left| \bigcap_{i \neq j} A_{i,j} \right|.$$

We will explicitly evaluate $|S|$ and $|A_{i,j}|$, as well as portions of $|A_{i,j} \cap A_{l,m}|$ and $|A_{i,j} \cap A_{l,m} \cap A_{o,p}|$. Then we will be able to show that $|S - (\bigcup A_{i,j})| > 0$ for particular cases of $n$ and $k$.

**Lemma 2.38.** Let $\omega_i, \omega_j$ be distinct. Then

$$\sum |A_{i,j}| = \binom{n}{k+1} \binom{k+1}{k} \frac{(n - k - 1) \left( \binom{n}{k+1} - 2 \right)}{2} \binom{n}{k+1} - 2.$$

**Proof.** There are $\binom{n}{k+1}$ sets $A_{i,j}$. However, many of these are the empty set. We need to count the number of nontrivial sets $A_{i,j}$. Each $\omega_i \in [k+1]^n$ has $\binom{k+1}{k}(n - k - 1)$ sets $\omega_j \in [k+1]^n$ such that $|\omega_i \cap \omega_j| = k$. There are $\binom{n}{k+1}$ sets in $[k+1]^n$. Since each $A_{i,j}$ will also appear as a set $A_{j,i}$, we multiply our count by $1/2$. Thus there are $\binom{n}{k+1} \binom{k+1}{k} (n - k - 1)/2$ nontrivial sets $A_{i,j}$.

If $|\omega_i \cap \omega_j| = k$, then

$$|A_{i,j}| = \binom{n}{k+1} \frac{\binom{n}{k+1} - 2}{k+1}.$$

To see this note that $\Omega \in A_{i,j}$ must contain $\omega_i$ and $\omega_j$ as well as $\binom{n}{k} / (k+1) - 2$ other elements from $[k+1]$. These other elements in $\Omega$ can be anything from
Thus the size of $A_{i,j}$ is the number of ways we can choose $\binom{n}{k}/(k+1) - 2$ from $\binom{n}{k+1} - 2$ elements. \hfill \qed

**Corollary 2.39.** Given $n$ and $k$, if $\binom{n}{k+1}/k+1 = 2$, then the number of distinct cycle sets is $|S| - \sum |A_{i,j}|$.

*Proof.* Since $\binom{n}{k}/(k+1) = 2$, every $\binom{n}{k}/(k+1)$ sized set $\Omega$ contains two elements. Because $\{\omega_i, \omega_j, \omega_m\} \not\subset \Omega$ for any $\binom{n}{k}/(k+1)$ sized set $\Omega$, we know, for $i, j, l, m$ distinct, that $A_{i,j} \cap A_{j,m} = \emptyset$ and $A_{i,j} \cap A_{l,m} = \emptyset$. Since any intersection of more than two $A_{i,j}$ sets must intersect one of these, we know that any nontrivial intersection $A_{i,j} \cap A_{j,m} \cap \cdots \cap A_{o,p}$ is empty. Thus by the principle of inclusion-exclusion, $|S - (\cup A_{i,j})| = |S| - \sum |A_{i,j}|$. \hfill \qed

**Example 2.40.** We can verify Proposition 2.28, Lemma 2.30, Lemma 2.38, and Corollary 2.39 by considering $n = 4$, and $k = 1$. We know that there are exactly three distinct cycle sets, namely $\Gamma_1 = \{1, 2\}, \{3, 4\}$, $\Gamma_2 = \{1, 3\}, \{2, 4\}$, and $\Gamma_3 = \{1, 4\}, \{2, 3\}$. Since there are only two elements in each $\Gamma_i$, we know for any collection $\{A_{i,j}\}$ with $|\{A_{i,j}\}| > 1$ that $\cap A_{i,j} = \emptyset$. Thus for $n = 4$ and $k = 1$, we must have

$$\frac{\binom{n}{2,\ldots,2}}{\binom{n}{k}/(k+1)!} = \left(\frac{\binom{n}{k+1}}{\binom{n}{k+1}}\right) - \frac{\binom{n}{k+1}\binom{k+1}{k}(n-k-1)}{2} \frac{\binom{n}{k+1} - 2}{\binom{n}{k+1} - 2}.$$  

Indeed,

$$\frac{\binom{4}{2}}{2!} = \frac{6}{2} - \frac{6(2)(2)}{2} \frac{4}{0} = 3.$$

**Example 2.41.** We can also employ Corollary 2.39 to prove that no cycle set exists for $k = 2$ and $n = 4$. Note that $\binom{n}{k}/(k+1)$ is well-defined since $\binom{4}{2}/3 = 2$. Then Corollary 2.39 tells us that the number of cycle sets $2\Gamma_4$ is equal to $|S| - \sum |A_{i,j}|$. Thus we have

$$\binom{n}{k+1} - \frac{\binom{n}{k+1}\binom{k+1}{k}(n-k-1)}{2} \frac{\binom{n}{k+1} - 2}{\binom{n}{k+1} - 2}$$

$$= \frac{4}{2} - \frac{4(3)(1)}{2} \frac{2}{0} = 0.$$  

This is not surprising since $k = 2$ and $n = 4$ fail the divisibility condition given in Lemma 2.14. In particular, $2 \nmid \binom{3}{1}$.  

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Lemma 2.42. If \( \omega_i, \omega_j, \) and \( \omega_m \) are distinct such that \( A_{i,j} \) and \( A_{j,m} \) are nonempty, then

\[
\sum |A_{i,j} \cap A_{j,m}| = \frac{(n)}{k+1} \left( \frac{(n+1)}{\binom{n}{k}} \right) \left( \frac{(n-k)}{k+1} - 3 \right).
\]

Example 2.43. Let \( n = 8, \) and \( k = 3. \) Then \( \omega_1 = \{1,2,3,4\}, \omega_2 = \{1,2,3,5\}, \) and \( \omega_3 = \{2,3,5,6\} \) are distinct sets such that \( A_{1,2} \) and \( A_{2,3} \) are nontrivial. Then every element \( \Omega \in A_{1,2} \cap A_{2,3} \) contains \( \omega_1, \omega_2, \) and \( \omega_3. \)

Proof. There are \( \binom{n}{k+1} (\binom{k+1}{k}) (n-k-1)/2 \) nontrivial sets \( A_{i,j}. \) We fix \( A_{i,j} \) and count the sets \( A_{j,m} \) that are nontrivial and satisfy \( \omega_i, \omega_j, \) and \( \omega_m \) distinct. Since each \( A_{i,j} \) will also appear as a set \( A_{j,m}, \) we multiply our count by \( 1/2. \)

Note that given distinct \( \omega_i, \omega_j, \omega_m \in \binom{n}{k+1} \) the sets \( A_{i,j} \cap A_{j,m} \) and \( A_{i,j} \cap A_{i,m} \) must both be counted. Note that \( \omega_i \) occurs in \( (k+1)(n-k-1) - 1 \) nontrivial pairs \( \{\omega_i, \omega_m\} \) for \( j \neq m. \) Similarly, \( \omega_j \) occurs in \( (k+1)(n-k-1) - 1 \) nontrivial pairs \( \{\omega_i, \omega_m\} \) for \( i \neq m. \) Thus there are \( 2[(k+1)(n-k-1) - 1] \) nontrivial sets \( A_{j,m} \) such that \( \omega_i, \omega_j, \) and \( \omega_m \) are distinct.

Since \( |\omega_i \cap \omega_j| = k \) and \( |\omega_j \cap \omega_m| = k, \) we have

\[
|A_{i,j} \cap A_{j,m}| = \binom{n}{k+1} - 3.
\]

To see this note that an \( \Omega \in A_{i,j} \cap A_{j,m} \) must contain \( \omega_i, \omega_j, \) and \( \omega_m. \) The other elements in \( \Omega \) can be anything from \( \binom{n}{k+1} \setminus \{\omega_i, \omega_j, \omega_m\}. \) Thus the size of \( A_{i,j} \cap A_{j,m} \) is the number of ways we can choose \( \binom{n}{k}/(k+1) - 3 \) from \( \binom{n}{k+1} - 3 \) elements. \( \square \)

Lemma 2.44. If \( \omega_i, \omega_j, \) and \( \omega_m \) are distinct such that \( |\omega_i \cap \omega_j \cap \omega_m| = k, \) then \( A_{i,j}, A_{j,m}, \) and \( A_{i,m} \) are nonempty and

\[
\sum |A_{i,j} \cap A_{j,m} \cap A_{i,m}| = \binom{n}{k} \left( \frac{n-k}{3} \right) \left( \binom{n}{k+1} - 3 \right).
\]

Example 2.45. Let \( n = 8, \) and \( k = 3. \) Then \( \omega_1 = \{1,2,3,4\}, \omega_2 = \{1,2,3,5\}, \) and \( \omega_3 = \{1,2,3,6\} \) are distinct sets such that \( A_{1,2}, A_{1,3}, \) and \( A_{2,3}, \)
are nontrivial and $|\omega_1 \cap \omega_2 \cap \omega_3| = 3$. Then every element $\Omega \in A_{i,2} \cap A_{1,3} \cap A_{2,3}$ contains $\omega_1, \omega_2$, and $\omega_3$.

**Proof.** There are $\binom{n}{k}$ possible values for $\omega_i \cap \omega_j \cap \omega_m$. By Proposition 2.26, there are $n - k$ sets $\omega_i$ such that $\delta \subset \omega_i$. Thus we are choosing $\omega_i, \omega_j$, and $\omega_m$ from these $n - k$ options.

Since $A_{i,j}, A_{j,m}$, and $A_{i,m}$ are nontrivial, we have

$$|A_{i,j} \cap A_{j,m} \cap A_{i,m}| = \binom{n}{k+1} - 3 \cdot \frac{n}{k+1} / (k+1) - 3.$$ 

To see this note that an $\Omega \in A_{i,j} \cap A_{j,m} \cap A_{i,m}$ must contain $\omega_i, \omega_j, \omega_m$. The other elements in $\Omega$ can be anything from $\binom{n}{k+1} \setminus \{\omega_i, \omega_j, \omega_m\}$. Thus the size of $A_{i,j} \cap A_{j,m} \cap A_{i,m}$ is the number of ways we can choose $\binom{n}{k} / (k+1) - 3$ from $\binom{n}{k+1} - 3$ elements.

**Lemma 2.46.** If $\omega_i, \omega_j$, and $\omega_m$ are distinct such that $|\omega_i \cap \omega_j \cap \omega_m| \neq k$ and $A_{i,j}, A_{j,m},$ and $A_{i,m}$ are nonempty, then

$$\sum |A_{i,j} \cap A_{j,m} \cap A_{i,m}| = \frac{n}{k+1} \binom{k+1}{k} \frac{(n-k-1)}{2} \binom{k+1}{k-1} \binom{n}{k+1} - 3.$$

**Example 2.47.** Let $n = 8$, and $k = 3$. Then $\omega_1 = \{1, 2, 3, 6\}, \omega_2 = \{1, 2, 3, 5\},$ and $\omega_3 = \{2, 3, 5, 6\}$ are distinct sets such that $A_{1,2}, A_{1,3},$ and $A_{2,3}$ are nontrivial, but $|\omega_1 \cap \omega_2 \cap \omega_3| \neq 3$. Notice that every element $\Omega \in A_{1,2} \cap A_{1,3} \cap A_{2,3}$ contains $\omega_1, \omega_2$, and $\omega_3$.

**Proof.** There are $\binom{n}{k+1} \binom{k+1}{k} (n-k-1)/2$ nontrivial sets $A_{i,j}$. We fix $A_{i,j}$ and count the sets $A_{i,m}, A_{j,m}$ that are nontrivial and satisfy $\omega_i, \omega_j, \text{ and } \omega_m$ distinct such that $|\omega_i \cap \omega_j \cap \omega_m| \neq k$. Note that there are $\binom{k+1}{k}$ possible sets for $\omega_j \cap \omega_m$ since we need $|\omega_j \cap \omega_m| = k$. However, since we also need $|\omega_i \cap \omega_j \cap \omega_m| \neq k$, we cannot have $\omega_j \cap \omega_m = \omega_j \cap \omega_j$. So in fact, there are $\binom{k+1}{k} - 1$ possible sets for $\omega_j \cap \omega_m$. Similarly, there are $\binom{k+1}{k} - 1$ possible sets for $\omega_i \cap \omega_m$. Since $A_{i,j}, A_{j,m},$ and $A_{i,m}$ are nontrivial, the choice of $\omega_j \cap \omega_m$ or $\omega_i \cap \omega_m$ fixes $\omega_m$. Thus there are $2(\binom{k+1}{k} - 1)$ ways to choose $A_{j,m}$ and $A_{i,m}$. Since set intersections are unordered, we multiply our count by $1/3!$.
Since $A_{i,j}, A_{j,m}$, and $A_{i,m}$ are nontrivial, we have

$$|A_{i,j} \cap A_{j,m} \cap A_{i,m}| = \left( \binom{n}{k+1} - 3 \right).$$

To see this note that an $\Omega \in A_{i,j} \cap A_{j,m} \cap A_{i,m}$ must contain $\omega_i, \omega_j, \text{ and } \omega_m$. The other elements in $\Omega$ can be anything from $[k+1] \setminus \{\omega_i, \omega_j, \omega_m\}$. Thus the size of $A_{i,j} \cap A_{j,m} \cap A_{i,m}$ is the number of ways we can choose $(n)/(k+1) - 3$ from $(k+1) - 3$ elements.

**Corollary 2.48.** Given $n$ and $k$, if $(n)/(k+1) = 3$, then the number of distinct cycle sets is

$$|S| - \sum |A_{i,j}| + \sum |A_{i,j} \cap A_{j,m}| - \sum |A_{i,j} \cap A_{j,m} \cap A_{i,m}|.$$

**Proof.** Since $(n)/(k+1) = 3$, every $(n)/(k+1)$ sized set $\Omega$ contains three elements. Thus all set intersections with a membership condition that $\Omega$ contains more than three elements must be empty. The intersection of four or more distinct sets $A_{i,j}$ always has the condition that $\Omega$ contains at least 4 elements.

To see this, assume $|\Omega| \leq 3$. If $\Omega = \{\omega_1, \omega_2, \omega_3\}$, then $\Omega \in A_{i,j}$ means $\{i, j\} \subset \{1, 2, 3\}$. Since $(3)/2 = 3$, there are exactly 3 distinct sets $A_{i,j}$ that contain $\Omega$.

We must count the cases where $A_{i,j} \cap A_{l,m}$ and $A_{i,j} \cap A_{l,m} \cap A_{o,p}$ require that $|\Omega| \leq 3$.

Consider $A_{i,j} \cap A_{l,m}$ for $i, j, l, m$ distinct. Then $|\Omega| \geq 4$ for all $\Omega \in A_{i,j} \cap A_{l,m}$. Thus $|A_{i,j} \cap A_{l,m}| = 0$. So we must count the cases $A_{i,j} \cap A_{j,m}$ for $i, j, m$ distinct.

Consider $A_{i,j} \cap A_{l,m} \cap A_{o,p}$ for not necessarily distinct indices. By our analysis above, this set only has the condition $|\Omega| \leq 3$ when the set has the form $A_{i,j} \cap A_{j,m} \cap A_{i,m}$ for $i, j, m$ distinct. We can have either $|\omega_i \cap \omega_j \cap \omega_m| = k$ or $|\omega_i \cap \omega_j \cap \omega_m| \neq k$.

Thus, by the principle of inclusion-exclusion, our formula accounts for $|S - (\cup A_{i,j})|$.

**Example 2.49.** We can verify Proposition 2.28, Lemma 2.30, Lemma 2.38, Lemma 2.42, Lemma 2.44, Lemma 2.46, and Corollary 2.48 by considering
\[ n = 6, \text{ and } k = 1. \text{ Since there are only three elements in a cycle set, we know that if an element of } \cap A_{i,j} \text{ must contain more than three distinct elements to satisfy the membership condition for each } A_{i,j}, \text{ then } \cap A_{i,j} = \emptyset. \text{ Thus for } n = 6 \text{ and } k = 1, \text{ we must have}
\]

\[
\frac{n}{(k+1)!} = \left( \frac{n}{k+1} \right) \left( \frac{n}{k+1} \right) (n-k-1) \left( \frac{n}{k+1} - 2 \right)
\]

\[
+ \frac{n}{(k+1)!} \left( \frac{n}{k+1} \right) \left( \frac{n}{k+1} \right) (n-k-1) \left( \frac{n}{k+1} - 2 \right)
\]

\[
- \left( \frac{1}{k+1} \right)^{n-k} \frac{n}{(k+1)!} \left( \frac{n}{k+1} \right) \left( \frac{n}{k+1} \right) (n-k-1) \left( \frac{n}{k+1} - 2 \right)
\]

Indeed,

\[
\frac{\binom{6}{2,2,2}}{3!} = \binom{6}{2} - \binom{6}{3} \left( \binom{6}{2} - 2 \right)
\]

\[
+ \frac{\binom{6}{2}(\binom{4}{1})}{2} \left[ \binom{6}{2} - 2 \right]
\]

\[
- \left( \binom{6}{3} \right) \left( \binom{6}{2} - 3 \right) = 15.
\]

**Example 2.50.** We can also employ Corollary 2.48 to prove that no cycle set exists for \( k = 4 \) and \( n = 6 \). Note that \( \binom{6}{k}/(k+1) \) is well-defined since \( \binom{6}{4}/(5) = 3 \). Then Corollary 2.48 tells us that the number of cycle sets \( 4\Gamma_6 \) is equal to

\[
\binom{6}{3} \left( \frac{6(5)(4)}{2} \right) \left[ \binom{6}{3} - 2 \right] - \left( \binom{6}{3} \right) \left( \binom{6}{3} - 3 \right) = 0.
\]

However, this is not surprising since \( k = 4 \) and \( n = 6 \) fails the divisibility condition given in Lemma 2.14. In particular, \( 4 \nmid \binom{5}{3} = 10 \).
2.3 Stable Ordering

The \( \left[ \binom{n}{k} \right] \) universal cycle problem has proven more challenging than many other universal cycle questions. One primary reason is the nature of sets; we are trying to create an ordered cycle from unordered objects. Using cycle sets, we will establish the notion of a stable ordering to address this challenge. Ultimately, this will allow us to create an Eulerian graph that we can use to prove the existence of a universal cycle for \( k \)-subsets of \( [n] \).

Since sets are unordered, we can write the set \( S = \{a, b, c, d, e, f\} \) many ways; \( \{a, b, c, d, e, f\}, \{f, b, c, a, e, d\}, \) and \( \{c, b, f, d, a, e\} \) are among the valid possibilities.

![Diagram of set S]

If we wanted \( S \) to appear in a universal cycle, we would have to express \( S \) as a 6 digit sequence. Potential candidates include abcdef, fbea, and cbfdae. We will employ an equivalence relation and equivalence classes in order to assign a label to sets.

**Definition 2.51.** Consider all length \( k \) sequences with distinct digits formed from \([n]\). We say the sequence \( x_1x_2\ldots x_k \) corresponds to the set \( \{x_1, x_2, \ldots, x_k\} \). Define an equivalence relation such that \( x_1x_2\ldots x_k \sim y_1y_2\ldots y_k \) if and only if \( \{x_1, x_2, \ldots, x_k\} = \{y_1, y_2, \ldots, y_k\} \). We say that \( x_1x_2\ldots x_k \) is equivalent to \( y_1y_2\ldots y_k \).

**Example 2.52.** The sequence 5612 corresponds to the set \( \{1, 2, 5, 6\} \). Note that the sequences 5126, 1265, and 1625 also correspond to \( \{1, 2, 5, 6\} \). Thus 5612 \( \sim \) 5126 \( \sim \) 1265 \( \sim \) 1625.

We can use this equivalence relation and consider equivalence classes \( [x_1x_2\ldots x_k] \) on the set of length \( k \) sequences with distinct digits from \([n]\). Equivalence classes will give us a concrete relation between sequences and sets.

**Definition 2.53.** Let \( x_1x_2\ldots x_k \) be a representative of equivalence class \( [x_1x_2\ldots x_k] \). Then \( x_1x_2\ldots x_k \) corresponds to the set \( \{x_1, x_2, \ldots, x_k\} \). Furthermore, by the definition of our equivalence relation each representative
Lemma 2.54. There is a one-to-one correspondence between \([\binom{n}{k}]\) and the equivalence classes of all length \(k\) sequences with distinct digits from \([n]\) under relation \(\sim\).

Proof. Let \(A\) be the set of equivalence classes of length \(k\) sequences with distinct digits from \([n]\) under \(\sim\). Let \(B = \binom{n}{k}\).

Let \(f : A \rightarrow B\) by \([x_1x_2...x_k] \mapsto \{x_1, x_2, \ldots, x_k\}\).

We claim that \(A\) and \(B\) have the same size. Note that \(|B| = |[\binom{n}{k}]|\), which is the familiar \(\binom{n}{k}\). An equivalence class in \(A\) is formed by choosing \(k\) distinct letters from \([n]\) and creating all possible words with those \(k\) letters. Thus \(|A|\) is also \(\binom{n}{k}\).

\((f : A \rightarrow B\) is injective\) We know that \([x_1x_2...x_k]\) is the only equivalence class that corresponds to \(\{x_1, x_2, \ldots, x_k\}\) because if a sequence \(z_1z_2\ldots z_k\) corresponds to \(\{x_1, x_2, \ldots, x_k\}\) then \(x_1x_2...x_k \sim z_1z_2\ldots z_k\) and \(z_1z_2\ldots z_k\) is contained in \([x_1x_2...x_k]\) by the definition of \(\sim\).

Any injective function between two finite sets of the same cardinality is also a surjection. Thus \(|A| = |B|\) and \(f\) injective means \(f\) is also surjective. Thus \(f\) is invertible and we can conclude that there is a one-to-one correspondence between \([\binom{n}{k}]\) and the equivalence classes on all length \(k\) sequences with distinct digits from \([n]\). \(\square\)

We want to employ this one-to-one relationship when assigning a label, or ordering, to a set.

Definition 2.55. An ordering \(\sigma\) of a set \(\{x_1, x_2, \ldots, x_k\}\) is an equivalence class representative from \([x_1x_2...x_k]\).

Let \(x_1x_2...x_k\) be a representative of \([x_1x_2...x_k]\). Any permutation of the \(k\) characters in \(x_1x_2...x_k\) will be in the equivalence class \([x_1x_2...x_k]\). Since \([x_1x_2...x_k]\) corresponds to \(\{x_1, x_2, \ldots, x_k\}\), only sequences containing exactly \(x_1, x_2, \ldots,\) and \(x_k\) appear in \([x_1x_2...x_k]\). Thus there exist \(k!\) sequences in each equivalence class.

Example 2.56. Let \(k = 3\) and \(n = 8\). We know that the set \(\{3, 5, 7\}\) has \(3!\) possible orderings because there are \(3!\) sequences in the equivalence class [357]. Thus possible orderings are 357, 375, 537, 573, 735, and 753.
By picking a representative from the equivalence class \([x_1x_2 \ldots x_k]\) we are able to write the set \(\{x_1, x_2, \ldots, x_k\}\) as an ordered object.

Recall that a de Bruijn Graph is a directed graph that represents overlaps between strings of characters. We can’t employ a de Bruijn graph to construct a universal cycle of \([n]^k\) because the graph contains too much information. For example, the de Bruijn graph of length 4 sequences on \([8]\) contains the sequences 1234 and 1324 in its edge set, but 1234 and 1324 are representatives from the same equivalence class under \(\sim\). Thus an Eulerian circuit in this de Bruijn graph would fail to be a universal cycle because it would contain the set \(\{1,2,3,4\}\) at least twice.

Instead of examining the de Bruijn graph, we will construct a transition graph \(T\) that has many of the properties that we like about de Bruijn graphs while only containing the information that we want in our universal cycle.

**Definition 2.57.** Let \(S\) be a set of orderings of the elements of \([n]^k\) such that exactly one ordering appears of each element \(\delta\). The transition graph \(T\) of \(S\) is a directed graph with vertex and edge set

\[
V(T) := \{x_1x_2 \ldots x_{k-1} \text{ and } x_2x_3 \ldots x_k : x_1x_2 \ldots x_{k-1}x_k \in S\}
\]

\[
E(T) := \{(x_1x_2 \ldots x_{k-1}, x_2x_3 \ldots x_k) : x_1x_2 \ldots x_{k-1}x_k \in S\}.
\]

In other words, for each ordering \(x_1x_2 \ldots x_k\) in \(S\), there are vertices \(x_1 \ldots x_{k-1}\), and \(x_2 \ldots x_k\) in our vertex set \(V(T)\). Then \((x_1x_2 \ldots x_{k-1}, x_2x_3 \ldots x_k)\), also written \(x_1x_2 \ldots x_k\), is a directed edge in \(T\) going from vertex \(x_1x_2 \ldots x_{k-1}\) to vertex \(x_2x_3 \ldots x_k\).

Notice that the edge set of the digraph \(T\) contains exactly one representative per equivalence class under \(\sim\) and the vertex set contains exactly the length \(k-1\) sequences from \([n]\) that are needed as endpoints. Thus \(T\) contains the exact information we want in a universal cycle.

**Definition 2.58.** Given \(\lambda \in \[k^n\]\), we say that vertex \(v \in V(T)\) is \(\lambda\)-equivalent if \(v\)'s label corresponds to \(\lambda\).

**Example 2.59.** Consider \(k = 3\) and \(n = 6\). Let \(S\) contain the lexicographic ordering of each \(\delta \in \lfloor \frac{k}{3}\rfloor\). So \(S\) is the set of orderings 123, 124, 125, 126, 134, 135, 136, 145, 146, 156, 234, 235, 236, 245, 246, 256, 345, 346, 356, and 456.

Then \(V(T) = \{12, 13, 14, 15, 23, 24, 25, 26, 34, 35, 36, 45, 46, 56\}\).
Figure 2.60. The directed graph $T$.

The digraph $T$ contains one edge per set in $\left[\frac{9}{3}\right]$. Unfortunately $T$ does not have $d^+(v) = d^-(v)$, a necessary condition for $T$ to be Eulerian.

For valid values of $n$ and $k$, our objective is to create a set $S$ of orderings of the elements of $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]$ such that $T$ is Eulerian. The following theorem is a generic result about any directed graph $D$. The result will guide our choice of an ordering for our particular transition graph $T$.

**Definition 2.61.** In a generic directed graph $D$, a cycle $C$ is a list of distinct vertices that can be arranged cyclically such that

1. if $u$ immediately proceeds $v$ in the cyclic list, then there is an edge with tail $u$ and head $v$, and

2. $uv$ appears at most one time as a consecutive subsequence in our cyclic list.

**Lemma 2.62.** Suppose $D$ is a directed graph. Let $\{C_i\}_{i \in I}$ be a set of disjoint directed cycles such that $D = \bigcup_i C_i$. If the underlying graph of $D$ is connected, then $D$ is Eulerian.

**Proof.** As we showed in Lemma 1.4, a directed graph $D$ is Eulerian if and only if $d^+(v) = d^-(v)$ for each vertex $v \in V(D)$ and the underlying graph is connected. By the hypothesis we know that the underlying graph is connected. Thus we have at most one nontrivial component. All that is left to check is that $d^+(v) = d^-(v)$ for each vertex $v \in V(D)$. Let $v \in V(D)$ occur in precisely the cycles $C_1, C_2, \ldots, C_t$. Recall that disjoint means if $e \in E(C_i)$, then
$e \notin E(C_j)$ for $i \neq j$. By definition, for $v \in V(C_i)$ there exists $e, e' \in E(C_i)$ so that $e$ is directed into $v$ and $e'$ is directed out of $v$. Thus $d^+(v) = j = d^-(v)$. Therefore $D$ satisfies both conditions and is Eulerian.

If we can construct a set of disjoint directed cycles $\{C_i\}_{i \in I}$ such that $T = \bigcup_i C_i$ and the underlying graph of $T$ is connected, then we have an Eulerian graph. Recall from chapter 1 that a universal cycle corresponds to an Eulerian circuit in a transition graph $T$. Thus an Eulerian $T = \bigcup_i C_i$ will prove that a universal cycle exists for $[n]$. Our first task is to find a way to organize the edges of a graph $T$ as disjoint directed cycles.

In the context of our transition graph $T$, edges represent overlap between vertices. Consider the graph $T$ with edges from an ordering of elements in $[n]$. In $T$, we’ll be looking at directed cycles $C = \langle x_1 x_2 \ldots x_{k+1} \rangle$. The edge labels are the length $k$ consecutive subsequences of $x_1 x_2 \ldots x_{k+1}$ read cyclically. The vertex labels are the length $k-1$ consecutive subsequences of $x_1 x_2 \ldots x_{k+1}$ read cyclically. By looking at cycles generated by sets of size $k + 1$, we will try to find an Eulerian transition graph $T$.

**Definition 2.63.** The sequence $x_0 x_1 \ldots x_{k-1} x_k$ generates the cycle

$$C = \langle x_0 x_1 \ldots x_{k-1} x_k \rangle.$$

**Example 2.64.** Consider the 5-cycles (12345) and (13542).

The cycle (12345) has edges 1234, 2345, 3451, 4512, and 5123. These class representatives correspond to sets $\{1, 2, 3, 4\}$, $\{2, 3, 4, 5\}$, $\{1, 3, 4, 5\}$, $\{1, 2, 4, 5\}$, and $\{1, 2, 3, 5\}$ respectively. These sets are precisely the size four sets of $[5]$.

Now consider the cycle (13542). The edges are 1354, 3542, 5421, 4213, and 2135. These class representatives correspond to sets $\{1, 3, 4, 5\}$, $\{2, 3, 4, 5\}$, $\{1, 2, 4, 5\}$, $\{1, 2, 3, 4\}$, and $\{1, 2, 3, 5\}$ respectively. As in the first cycle, these sets are precisely the size four sets of $[5]$.

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Notice that the ordering of a cycle doesn’t affect the subsets that correspond to the edges of that cycle.

This example inspires an important fact about this type of $k + 1$ cycle.

**Lemma 2.65.** Given a set $\gamma_i \in [k+1] \cup \{n\}$ and any ordering $\sigma_i = x_0x_1 \ldots x_{k-1}x_k$ of $\gamma_i$, consider the cycle generated by the ordering of $\gamma_i$, namely $C_{\sigma_i} = (x_0x_1 \ldots x_{k-1}x_k)$. There exists a one-to-one correspondence between the edges in $C_{\sigma_i}$ and the sets $\delta \subset \gamma_i$ for $\delta \in [k]$. 

**Proof.** Let $A = \{\delta \in [k] : \delta \subset \gamma_i\}$. Let $B$ be the edges of the cycle $C_{\sigma_i} = (x_0x_1 \ldots x_{k-1}x_k)$ generated by an ordering $\sigma_i$ on $\gamma_i$.

Let $f : A \rightarrow B$ by \{\(x_1, x_2, \ldots, x_k\} \mapsto y_1y_2 \ldots y_k \in [x_1x_2 \ldots x_k]$.

We claim that $A$ and $B$ have the same size. Note that $|A| = \binom{k+1}{k} = k + 1$ because we are taking all $k$-subsets of the $k + 1$ integers in $\gamma_i$. Since $C_{\sigma_i}$ is a cycle generated by a length $k + 1$ sequence, we know that there are a total of $k + 1$ length $k$ consecutive subsequences and thus $k + 1$ edges in $C$ and $|B| = k + 1$.

$(f : A \rightarrow B$ is injective$).$ By the definition of $\sim$ we know that edge $e = x_1x_2 \ldots x_k$ corresponds to exactly one $\delta \in [k]$, namely $\delta = \{x_1, x_2, \ldots, x_k\}$. Thus there exists no distinct $\delta_1, \delta_2 \in [k]$ with $\delta_1 \subset \gamma_i$ and $\delta_2 \subset \gamma_i$ such that $e$ corresponds to $\delta_1$ and to $\delta_2$.

Thus there is a one-to-one correspondence between the edges in $C_{\sigma_i}$ and the sets $\delta \subset \gamma$. \qed

We want to apply this result to any ordering of the elements in a cycle set such that the ordering generates a collection $\{C_{\sigma_i}\}_{i \in I}$ of disjoint directed cycles. Further, by applying an ordering to every element of a cycle set, we can establish a correspondence between the edges appearing in $E(\cup_i C_{\sigma_i})$ and the elements of $[n]$. 

**Definition 2.66.** Let $\Gamma$ be a cycle set. Consider an ordering $\sigma_i$ of each $\gamma_i \in \Gamma$. Then the set of orderings $\{\sigma_i\}$ is an **ordering on** $\Gamma$.

**Example 2.67.** Consider the following elements from a cycle set $\Gamma_8$. We can arbitrarily assign an ordering to these elements and consider the cycles generated.
<table>
<thead>
<tr>
<th>Element</th>
<th>Ordering</th>
</tr>
</thead>
<tbody>
<tr>
<td>{1, 2, 3, 4}</td>
<td>1234</td>
</tr>
<tr>
<td>{1, 2, 7, 8}</td>
<td>1728</td>
</tr>
<tr>
<td>{1, 3, 6, 7}</td>
<td>1763</td>
</tr>
<tr>
<td>{1, 4, 6, 8}</td>
<td>1468</td>
</tr>
<tr>
<td>{2, 3, 6, 8}</td>
<td>2386</td>
</tr>
</tbody>
</table>

The cycles (1234), (1728), (1763), (1468), and (2386).

Consider the set \{1, 3, 4\} that appears as a subset of \{1, 2, 3, 4\}. Notice that \{1, 3, 4\} does not appear as a subset of an element other than \{1, 2, 3, 4\}. The edge 341 corresponds to this set and appears in the cycle generated by the ordering on \{1, 2, 3, 4\}. This is the only edge in the collection of cycles that corresponds to \{1, 3, 4\}.

This is true for any \(\delta \in {\mathbb{N}}^{3}\) that appears as a subset of one of our listed elements. Based on the previous lemma we are allowed to generalize this result to any cycle set.

**Corollary 2.68.** Given a cycle set \(\Gamma\) and an ordering \(\{\sigma_i\}\) on \(\Gamma\), let \(C_{\sigma_i}\) be the cycle \(C_{\sigma_i} = (x_0 x_1 \ldots x_k)\) generated by ordering \(\sigma_i = x_0 x_1 \ldots x_k\) for \(\gamma_i = \{x_0, x_1, \ldots, x_k\}\). There exists a one-to-one correspondence between the edges \(e \in E(\cup_i C_{\sigma_i})\) and the set \([n/k]\).

**Proof.** Let \(A\) be the edges in the set \(E(\cup_i C_{\sigma_i})\). Let \(B = [n/k]\).

Let \(f : A \to B\) by \(x_1 x_2 \ldots x_k \mapsto \{x_1, x_2, \ldots x_k\}\).

\((f : A \to B\) is surjective.) Consider an arbitrary \(\delta \in [n/k]\). By definition of a cycle set, there exists \(\gamma_i \in \Gamma\) such that \(\delta \subset \gamma_i\). Let \(\sigma_i\) be the ordering of \(\gamma_i\). By the previous lemma we know that there exists a one-to-one correspondence between the edges in the cycle \(C_{\sigma_i}\) generated by \(\sigma_i\) and the \(k\)-subsets of \(\gamma_i\). Thus we know that there exists an \(e \in E(C_{\sigma_i})\) such that \(f(e) = \delta\).
Recall that the definition of a cycle set states that for every $\delta \in [\frac{n}{k}]$ there exists a unique $\gamma \in \Gamma$ with $\delta \subset \gamma$. Assume for a contradiction that there are distinct edges $e_1, e_2 \in E(\bigcup_i C_{\sigma_i})$ such that $e_1 \sim e_2$. In other words, $e_1$ and $e_2$ are distinct edges that correspond to the same $\delta$. By the previous lemma we know that $e_1$ and $e_2$ cannot be in the same cycle. So there exist distinct $C_{\sigma_1}, C_{\sigma_2}$ such that $e_1 \in E(C_{\sigma_1})$ and $e_2 \in E(C_{\sigma_2})$. Then $C_{\sigma_1}$ corresponds to some $\gamma_1 \in \Gamma$ and $C_{\sigma_2}$ corresponds to some $\gamma_2 \in \Gamma$. Since $C_{\sigma_1} \neq C_{\sigma_2}$ we know that $\gamma_1 \neq \gamma_2$. Since $e_1$ and $e_2$ are distinct edges that correspond to the the same $\delta$ we can conclude that $\delta \subset \gamma_1$ and that $\delta \subset \gamma_2$. This contradicts the definition of a cycle set.

By applying some ordering on $\Gamma$, we have arranged the edges that correspond to sets $\delta \in [\frac{n}{k}]$ in cycles. If we can find an ordering such that $\bigcup C_{\sigma}$ is connected, then our digraph $T$ is Eulerian and we have proven the existence of a universal cycle for $[\frac{n}{k}]$.

**Definition 2.69.** Let $\{C_{\sigma_i}\}$ be the set of cycles of the form $C_{\sigma_i} = (x_0x_1 \ldots x_k)$ generated by an ordering $\{\sigma_i\}$ on cycle set $\Gamma$. A stable ordering of cycle set $\Gamma$ is an ordering such that the underlying graph of $\bigcup C_{\sigma_i}$ is connected.

### 2.4 Stable Ordering Existence

#### 2.4.1 A Heuristic

**Figure 2.70.** We have a process for assigning an ordering to a cycle set $\Gamma$.

<table>
<thead>
<tr>
<th>Step</th>
<th>Instruction</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Choose an arbitrary $\gamma_i \in \Gamma$. Assign any ordering $\sigma_i$ to $\gamma_i$. If $\binom{n}{k}/(k+1) = 1$, stop. Otherwise, proceed to step 2.</td>
</tr>
<tr>
<td>2</td>
<td>Consider $C_{\sigma_i}$. There is a $\lambda_i$-equivalent vertex in $C_{\sigma_i}$ such that there is an unordered $\gamma_j \in \Gamma$ with $\lambda_i \subset \gamma_j$. Arbitrarily choose one such $\gamma_i$ and proceed to step 3.</td>
</tr>
<tr>
<td>3</td>
<td>Assign an ordering to this unordered $\gamma_j$ such that I. The set of cycles ${C_{\sigma_i}}$ generated by the ordered elements of $\Gamma$ creates a connected digraph $\bigcup C_{\sigma_i}$. II. We maximize the number of vertices occurring in $V(\bigcup C_{\sigma_i})$. If a maximum occurs in multiple ways, arbitrarily choose one ordering. Proceed to step 4.</td>
</tr>
</tbody>
</table>
### Step 4 Instruction

Consider \( \{ C_{\sigma} \} \) the set of cycles generated by the ordered elements of \( \Gamma \). If there is a \( \lambda_j \)-equivalent vertex in \( \{ C_{\sigma} \} \) such that there exists an unordered \( \gamma \in \Gamma \) with \( \lambda_j \subset \gamma \), then proceed to step 3 with \( \lambda_j \). If multiple such \( \lambda_j \) exist, then arbitrarily choose one and proceed to step 3. If no such \( \lambda \) exists, then proceed to step 5.

### Step 5 Instruction

If there exists an unordered \( \gamma \in \Gamma \), then discard \( \{ \sigma_i \} \). Proceed to step 1 to start over. Otherwise, proceed to step 6.

### Step 6 Instruction

If every \( \gamma \in \Gamma \) has been assigned an ordering, then stop.

---

**Flowchart of Process 2.70**

\[
1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow \text{end}
\]

One can easily see the following.

**Lemma 2.71.** If Process 2.70 terminates, then the ordering of cycle set \( \Gamma \) is stable.

Unfortunately, it is unclear if Process 2.70 terminates. However, \( k = 2 \) provides a unique case.

**Example 2.72.** Consider \( n = 7 \) and \( k = 2 \). Given cycle set \( \Gamma \), we will use the process to construct a stable ordering.

<table>
<thead>
<tr>
<th>( \overset{2}{\Gamma}_7 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>{1, 2, 4}, {1, 5, 7}, {1, 3, 6}, {2, 5, 6}</td>
</tr>
<tr>
<td>{2, 3, 7}, {3, 4, 5}, {4, 6, 7}</td>
</tr>
</tbody>
</table>

Step one: Consider \{3, 4, 5\}. Assign ordering 354. Consider the cycle (354) generated by this ordering.

![Step two diagram]

Step two: We have vertex 3 which is equivalent to \{3\}. There are unordered \( \gamma \in \overset{2}{\Gamma}_7 \) such that \{3\} appears as a subset of \( \gamma \).

Step three: We want to assign an ordering to \{1, 3, 6\}, and \{2, 3, 7\}. For \( k = 2 \), our vertices are single numbers. Thus any ordering maximizes the
number of $\lambda$-equivalent vertices appearing. Assign ordering 136 and 273. Consider the graph generated by cycles \{$(354), (136), (273)$\}.

Step four: Notice that vertex 7 is equivalent to \{7\}. Since there are unordered elements in $\Gamma$ that contain \{7\} as a subset, we will proceed to step three.

Step three: We want to assign an ordering to \{1, 5, 7\}, and \{4, 6, 7\}. Assign ordering 157 and 476. Consider the graph generated by cycles

\{$(354), (136), (273), (157), (476)$\}.

Step four: Notice that vertex 2 is equivalent to \{2\}. Since there are unordered elements in $\Gamma$ that contain \{2\} as a subset, we will proceed to step three.

Step three: We want to assign an ordering to \{1, 2, 4\}, and \{2, 5, 6\}. Assign ordering 142 and 256. Consider the graph generated by cycles

\{$(354), (136), (273), (157), (476), (142), (256)$\}.
Step five/six: Since every element in $\Gamma$ has been assigned an ordering, we know by Lemma 2.71 that our ordering $\{354, 136, 273, 157, 476, 142, 256\}$ is stable.

In general, 2.70 will terminate for $k = 2$. In fact, we have an even stronger result.

**Lemma 2.73.** Given a cycle set $^2\Gamma_n$, any ordering of $\Gamma$ is stable.

**Proof.** Consider an arbitrary ordering $\{\sigma_i\}$ on $^2\Gamma_n$ and the set of cycles $\{C_{\sigma_i}\}$ generated by that ordering. Since $k = 2$, vertices in $\{C_{\sigma_i}\}$ are single integers. Thus every $\lambda \in \{n\}$ appears as a vertex in $\{C_{\sigma_i}\}$. Consider the digraph $\cup C_{\sigma_i}$. We must show that the underlying graph $G$ is connected.

Let $\{a\}, \{b\} \in \{n\}$. Then there are vertices $a, b \in V(G)$. We also know that $\{a, b\} \in \{\frac{n}{2}\}$. Thus by definition of cycle set, there exists some element $\gamma \in ^2\Gamma_n$ such that $\{a, b\} \subset \gamma$. Thus there is some cycle in $\{C_{\sigma_i}\}$ that contains directed edge $ab$ or edge $ba$. Thus we have $ab \in E(G)$. \qed

**Example 2.74.** Consider $k = 2$ and $n = 7$. We have a valid cycle set $^2\Gamma_7$ with three distinct orderings. (There are $2^7$ possible orderings of this set.)

<table>
<thead>
<tr>
<th>$^2\Gamma_7$</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>${1, 2, 3}$</td>
<td>(123)</td>
<td>(132)</td>
<td>(123)</td>
</tr>
<tr>
<td>${1, 4, 5}$</td>
<td>(145)</td>
<td>(145)</td>
<td>(145)</td>
</tr>
<tr>
<td>${1, 6, 7}$</td>
<td>(167)</td>
<td>(176)</td>
<td>(167)</td>
</tr>
<tr>
<td>${2, 4, 6}$</td>
<td>(246)</td>
<td>(264)</td>
<td>(264)</td>
</tr>
<tr>
<td>${2, 5, 7}$</td>
<td>(257)</td>
<td>(275)</td>
<td>(257)</td>
</tr>
<tr>
<td>${3, 4, 7}$</td>
<td>(347)</td>
<td>(374)</td>
<td>(347)</td>
</tr>
<tr>
<td>${3, 5, 6}$</td>
<td>(356)</td>
<td>(356)</td>
<td>(365)</td>
</tr>
</tbody>
</table>
Indeed, regardless of the ordering, the directed graph $T_i$ is connected.

We will treat Process $2.70$ as a heuristic. In the next section, we explore the likelihood that a stable ordering exists for particular values of $n$ and $k$.

### 2.4.2 Probability

Given $\lambda \in \left[ \binom{n}{k-1} \right]$, it is not necessary for a $\lambda$-equivalent vertex to appear in the set of cycles generated from an ordering $\{\sigma_i\}$ on a cycle set $\Gamma$ for that ordering to be stable. However, consider step 4 of our heuristic (Process $2.70$). We look for a $\lambda$-equivalent vertex in $\{C_\sigma\}$ such that there exists an unordered $\gamma \in \Gamma$ with $\lambda \subset \gamma$. Our goal is to assign an ordering to every unordered element in $\Gamma$ such that the resulting generated cycle graph $\bigcup C_{\sigma_i}$ is connected. Increasing the number of $\lambda$-equivalent vertices appearing in the set of generated cycles gives us a greater potential for establishing connectedness.

To see this, assume that we have an ordering on certain elements of a cycle set $\Gamma$ such that the cycles generated from the ordering are connected and $\lambda$-equivalent vertices appear for a significant number of $\lambda \in \left[ \binom{n}{k-1} \right]$. Our goal is to label the unordered elements of $\Gamma$ such that the generated cycles are connected in our graph. The probability of finding a vertex to connect a cycle to increases when we increase the variety of vertices available to connect to.

Thus, we will study the occurrences of $\lambda$-equivalent vertices in arbitrary orderings.

**Example 2.75.** Let $k = 3$ and $n = 8$. Consider the set $\gamma = \{1, 2, 3, 4\}$. There are $4!$ possible orderings for $\gamma$. We want to know how many cycles generated from these orderings have a vertex $v$ that is equivalent to $\{1, 2\}$. There are $2!$ orderings of $\{1, 2\}$, namely $12$ and $21$. Thus we want to know
which orderings of \( \gamma \) have 12 or 21 as a consecutive subsequence. Note that we should read the orderings of \( \gamma \) cyclically because we are employing each ordering as a cycle. For example, the ordering 1342 generates the cycle (1342) which has the vertex 21.

Furthermore, notice that a sequence and its cyclic shifts each generate the same cycle. For instance, the sequences 1234, 2341, 3412, and 4123 all generate the cycle (1234). Thus if we list the orderings of \( \gamma \) that have 1 and 2 in the first two positions, we have also listed all cycles that have a vertex equivalent to \( \{1, 2\} \). Therefore cycles that contain a vertex equivalent to \( \{1, 2\} \) are (1234), (1243), (2134), and (2143).

There are two cycles generated by an ordering of \( \{1, 2, 3, 4\} \) that do not contain a \( \{1, 2\} \) equivalent vertex. They are (1324) and (1423).

Then the total number of cycles that can be created from orderings of \( \gamma \) is 6. There are 4 orderings that generate each cycle. Thus if we pick an ordering of \( \gamma \) with equal probability and consider \( C \) the cycle generated by our choice of ordering, then \( P(C \text{ has a } \{1, 2\} \text{-equivalent vertex}) = 4/6 \).

We can generalize several facts from this example. We will then compute the probability that a \( \lambda \)-equivalent vertex does not appear as a vertex of a cycle.
Lemma 2.76. There are \( k+1 \) sequences that generate the cycle \( C = (x_0 \ldots x_k) \).

Proof. There are \( k + 1 \) distinct expressions for cycle \( C \), namely
\[
(x_0 x_1 \ldots x_k) = (x_1 x_2 \ldots x_0) = \ldots = (x_k x_0 \ldots x_{k-1}).
\]
Thus \( x_0 x_1 \ldots x_k, x_1 x_2 \ldots x_0, \ldots \) and \( x_k x_0 \ldots x_{k-1} \) generate \( C \). \( \Box \)

This proof also demonstrates the following corollary.

Corollary 2.77. If an ordering \( \sigma \) generates the cycle \( C_\sigma \), then \( \sigma \)'s cyclic shifts also generate \( C_\sigma \).

We use these facts to calculate certain probabilities.

Lemma 2.78. Let \( \lambda \in [k^{-1}] \) and \( \gamma \in [k+1] \) such that \( \lambda \subset \gamma \). If we pick an ordering \( \sigma \) of \( \gamma \) with equal probability and consider the cycle \( C_\sigma \) generated by our choice \( \sigma \), then
\[
P(C_\sigma \text{ has a } \lambda\text{-equivalent vertex}) = \frac{2}{k}.
\]

Proof. Let \( \lambda \in [k^{-1}] \) and \( \gamma \in [k+1] \) such that \( \lambda \subset \gamma \). Then \( |\gamma| = k + 1 \) and \( |\lambda| = k - 1 \).

First, we will count all of the cycles \( C_\sigma \) such that \( C_\sigma \) has a \( \lambda \)-equivalent vertex. We want the \( k - 1 \) elements of \( \lambda \) to occur consecutively in a cycle. Since the cyclic shifts of an ordering all generate the same cycle, we can always write the \( k - 1 \) elements of \( \lambda \) first and follow them with the two elements of \( \gamma \setminus \lambda \). There are \( (k - 1)! \) distinct vertices equivalent to \( \lambda \). There are \( 2! \) ways to arrange the other two elements. Thus there are \( (k - 1)!2! \) distinct cycles with a \( \lambda \)-equivalent vertex.

Now we will count the number of cycles \( C_\sigma \) that correspond to \( \gamma \). There are \( (k + 1)! \) orderings of \( \gamma \). Since each cycle is generated by \( k + 1 \) orderings of \( \gamma \), we need to divide by \( k + 1 \). Thus there are \( k! \) distinct cycles that correspond to \( \gamma \).

Therefore, \( P(C_\sigma \text{ has a } \lambda\text{-equivalent vertex}) = \frac{(k-1)!2!}{k!} = \frac{2}{k}. \) \( \Box \)

As a consequence, for small values of \( k \), the probability of a vertex appearing in a cycle generated from an ordering is high.
Example 2.79. For $k = 2$, we can use Lemma 2.73 to verify Lemma 2.78. We saw in 2.73 that if $\lambda \subset \gamma$, then a $\lambda$-equivalent vertex must appear in any ordering of $\gamma$. Thus we must have $P(C_\sigma$ has a $\lambda$-equivalent vertex) $= 1$. Indeed $2/k = 1$ for $k = 2$.

We easily use Lemma 2.78 to find the probability that a $\lambda$-equivalent vertex does not appear in an ordering of a cycle.

Corollary 2.80. Let $\lambda \in \lfloor \frac{n}{k-1} \rfloor$ and $\gamma \in \lfloor \frac{n}{k+1} \rfloor$ such that $\lambda \subset \gamma$. If we pick an ordering $\sigma$ of $\gamma$ with equal probability and consider the cycle $C_\sigma$ generated by our choice $\sigma$, then $P(C_\sigma$ does not have a $\lambda$-equivalent vertex) $= \frac{k-2}{k}$.

Proof. By the previous result, $P(C_\sigma$ has a $\lambda$-equivalent vertex) $= \frac{2}{k}$. Thus $P(C_\sigma$ does not have a $\lambda$-equivalent vertex) $= 1 - \frac{2}{k} = \frac{k-2}{k}$.

As $k$ becomes large, the probability that a vertex appears in a cycle generated by an ordering is low. Further, this probability drops quite quickly. For $k = 12$, we have a probability of approximately 0.167. *Prima facie*, this seems to indicate that a stable ordering is improbable.

However, note that the probability a vertex appears in a generated cycle is independent of $n$. We will show that the number of cycles a $\lambda$-equivalent vertex can potentially appear in increases with the size of $n$, and thus, for $n$ large and $k$ relatively small, we have higher probability of $\lambda$ appearing as a vertex.

Lemma 2.81. Consider a cycle set $\Gamma$ and randomly assign an ordering $\{\sigma_i\}$ on $\Gamma$. Let $\{C_\sigma\}$ be the set of cycles generated by $\{\sigma_i\}$. The ratio of vertices appearing in the generated cycles (allowing for repetition) to potential vertex labelings is $\frac{n-k+1}{k!}$.

Proof. There are $\binom{n}{k}$ cycles generated by the ordering on $\Gamma$. Since there are $k+1$ vertices appearing in each cycle, we have $\binom{n}{k}$ vertices (not necessarily distinct) appearing in the cycles generated by the ordering on $\Gamma$. We know that there are $\binom{n}{k-1}$ elements in $\lfloor \frac{n}{k-1} \rfloor$. Thus there are $(k-1)!\binom{n}{k-1}$ potential vertices that can appear in the cycles generated by the ordering on $\Gamma$.

Consider the ratio $\frac{\binom{n}{k}}{(k-1)!\binom{n}{k-1}}$. This simplifies to $\frac{n-k+1}{k!}$.

We have two corollaries.
Corollary 2.82. The ratio of vertices appearing in the generated cycles (allowing for repetition) to potential vertex labelings increases as $n$ increases and for $(n - k + 1) > k!$ this ratio is greater than 1.

Corollary 2.83. For $k$ fixed, as $n \to \infty$ the ratio of vertices appearing in the generated cycles (allowing for repetition) to potential vertex labelings is

$$\lim_{n \to \infty} \frac{n - k + 1}{k!} = \infty.$$  

As we already noted, it is not necessary for every potential vertex labeling to appear in the cycles generated from an ordering on cycle set $\Gamma$. We primarily want at least one $\lambda$-equivalent vertex to appear in the generated cycles rather than all $\lambda$-equivalent vertices for $\lambda \in \left[\frac{n}{k} - 1\right]$. Thus our situation is better than the ratio indicates.

Again, consider a cycle set $\Gamma$ and randomly assign an ordering $\{\sigma_i\}$ on $\Gamma$. Let $\{C_\sigma\}$ be the set of cycles generated by $\{\sigma_i\}$. We can analyze the potential to create a stable ordering by considering the probability of being able to create fewer components in the case that the generated cycles are disconnected. We begin by assuming that one component in the graph of generated cycles is a single cycle.

Lemma 2.84. Consider a cycle set $\Gamma$ and randomly assign an ordering $\{\sigma_i\}$ on $\Gamma$. Let $\{C_\sigma\}$ be the set of cycles generated $\{\sigma_i\}$. Assume that there is a cycle $C_{\sigma_1} \in \{C_\sigma\}$ such that $C_{\sigma_1}$ is disconnected from every other cycle. Let $C_{\sigma_1}$ correspond to $\gamma_1$. Then $P(\gamma_1$ cannot be reordered such that $C_{\sigma_1}$ is connected to a cycle in $\{C_\sigma\} \setminus C_{\sigma_1})$

$$= \left(\frac{k - 2}{k}\right)^{\frac{n - k - 1}{2} \binom{k + 1}{k - 1}}.$$  

To prove this we need a proposition.

Proposition 2.85. Let $\Gamma$ be a cycle set. If $\lambda_1, \lambda_2 \in \left[\frac{n}{k} - 1\right]$ are distinct subsets of $\gamma_1 \in \Gamma$ then there is no $\gamma_j \in \Gamma \setminus \{\gamma_1\}$ such that $\lambda_1 \subset \gamma_j$ and $\lambda_2 \subset \gamma_j$.

Proof. To see this, consider the fact that $\lambda_1$ and $\lambda_2$ are distinct. Thus they share at most $k - 2$ elements in common. Thus $|\lambda_1 \cup \lambda_2| \geq k$. Notice that $|\lambda_1 \cup \lambda_2| \leq k + 1$ because $\lambda_1 \cup \lambda_2$ is contained inside the size $k + 1$ set $\gamma_1$. Then we have two cases.
First case: Let $|\lambda_1 \cup \lambda_2| = k$. If $\lambda_1$ and $\lambda_2$ both appear in $\gamma_1$ and in $\gamma_j$, then $\delta = \lambda_1 \cup \lambda_2 \in \mathbb{[n]}$ satisfies $\delta \subset \gamma_1$ and $\delta \subset \gamma_j$, which contradicts the definition of a cycle set.

Second Case: $|\lambda_1 \cup \lambda_2| = k + 1$. If $\lambda_1$ and $\lambda_2$ both appear in $\gamma_1$ and in $\gamma_j$, then $\gamma_1 = \gamma_j$. This contradicts the assumption that $\gamma_1$ and $\gamma_j$ are distinct.

Now we can prove Lemma 2.84.

**Proof.** There are \(\binom{k+1}{k-1}\) distinct $k-1$-subsets $\lambda$ contained in $\gamma_1$. Each set $\lambda$ occurs in $n-(k-1)$ sets $\delta \in \mathbb{[n]}$. Note that if $\lambda \subset \gamma \in \Gamma$ then $\gamma = \lambda \cup \{a, b\}$ for some distinct $a, b \in [n]$. Thus there exists distinct $\delta_1 = \lambda \cup \{a\}, \delta_2 = \lambda \cup \{b\}$ such that $\delta_1 \subset \gamma_1$ and $\delta_2 \subset \gamma_1$. Thus each $\lambda$ occurs in $(n-k+1)/2$ elements of a given cycle set. We know that $\lambda$ occurs in $\gamma_1$. So there are $(n-k-1)/2$ sets in $\Gamma \setminus \{\gamma_1\}$ that contain $\lambda$.

By Proposition 2.85 we can conclude that if a $\lambda_1$-equivalent vertex can appear in a cycle generated from an ordering $\sigma_j$ of $\gamma_j \in \Gamma \setminus \{\gamma_1\}$, then a $\lambda_2$-equivalent vertex cannot appear in any ordering of $\gamma_j$.

Similarly, if a $\lambda_2$-equivalent vertex can appear in a cycle generated from an ordering $\sigma_j$ of $\gamma_j \in \Gamma \setminus \{\gamma_1\}$, then a $\lambda_1$-equivalent vertex cannot appear in any ordering of $\gamma_j$.

Recall by Corollary 2.80 that the probability a cycle does not have a $\lambda$-equivalent vertex for $\lambda \in \mathbb{[n]}$ is $(k-2)/k$. Thus for each $\lambda \subset \gamma_1$, the probability that a $\lambda$-equivalent vertex does not appear in a cycle of $\{C_\sigma\} \setminus C_{\sigma_1}$ is $\left(\frac{k-2}{k}\right)^{\frac{n-k-1}{2}(k+1)}$. For each $\lambda \subset \gamma_1$, we know that $\lambda$-equivalent vertices appear independently in an ordering on $\Gamma \setminus \{\gamma_1\}$. Thus we raise this probability to the $\binom{k+1}{k-1}$ power. \(\square\)

**Corollary 2.86.** For $k$ fixed and $n \to \infty$, $P(\gamma_1 \text{ cannot be reordered such that } C_{\sigma_1} \text{ is connected to a cycle in } \{C_\sigma\} \setminus C_{\sigma_1})$

$$= \lim_{n \to \infty} \left(\frac{k-2}{k}\right)^{\frac{n-k-1}{2}(k+1)} = 0.$$

We can also show that for a sufficient number of disconnected cycles, there must be some reordering such that the graph $T = \cup_i C_{\sigma_i}$ has fewer components.

**Lemma 2.87.** Consider a cycle set $\Gamma$ and randomly assign an ordering to the elements $\gamma \in \Gamma$. Let $\{C_\sigma\}$ be the set of cycles generated by the ordering
Assume that there is a set of cycles $M = \{C_{\sigma_1}, C_{\sigma_2}, \ldots, C_{\sigma_m}\} \subseteq \{C_{\sigma}\}$ such that each $C_{\sigma_i} \in M$ is disconnected from every cycle in $\{C_{\sigma}\} \setminus M$. Let $C_{\sigma_i}$ correspond to $\gamma_i$.

If $\binom{k+1}{k-1} m - \binom{m}{2} \geq \binom{n}{k-1}$, then $P(\gamma_1, \ldots, \gamma_m)$ cannot be reordered such that some $C_{\sigma_i} \in M$ is connected to a cycle in $\{C_{\sigma}\} \setminus M$.

Proof. Recall Lemma 2.7 states for $i \neq j$ we have $|\gamma_i \cap \gamma_j| \leq k - 1$. Vertices correspond to size $k - 1$ sets $\lambda$. For $\gamma_i, \gamma_j$ distinct, let $C_{\sigma_i}$ be generated by an ordering of $\gamma_i$ and let $C_{\sigma_j}$ be generated by an ordering of $\gamma_j$. We know that $C_{\sigma_i}$ and $C_{\sigma_j}$ share at most one vertex. We claim that the number of distinct $k - 1$-subsets $\lambda$ among elements $\gamma_i \in \{\gamma_1, \ldots, \gamma_m\}$ is at least $\binom{k+1}{k-1} m - \binom{m}{2}$. To see this note that each $\gamma_i$ has $\binom{k+1}{k-1}$ $k - 1$-subsets $\lambda$. There are $m$ elements $\gamma_i$ and we know that between any two elements $\gamma_i, \gamma_j$ we have at most one size $k - 1$ set $\lambda$ in common.

Thus if $\binom{k+1}{k-1} m - \binom{m}{2} \geq \binom{n}{k-1}$, we know that the distinct $k - 1$-subsets of elements in $\gamma_1, \ldots, \gamma_m$ are $\left[\frac{n}{k-1}\right]$. Therefore if $\{C_{\sigma}\} \setminus M \neq \emptyset$, then there is some $\lambda \in \left[\frac{n}{k-1}\right]$ such that a $\lambda$-equivalent vertex appears in $\{C_{\sigma}\} \setminus M$. □

Thus, while we can’t explicitly prove the existence of a stable ordering for a cycle set, we have reason to believe that one exists, particularly for $n$ large and $k$ relatively small.
2.5 Result

Given the existence of the machinery we have defined, we have an intermediate step that allows us to employ the classic existence approaches to prove that a universal cycle exists.

**Theorem 2.88.** Given a cycle set $k\Gamma_n$ with a stable ordering, there exists a universal cycle of $k$-subsets of $[n]$.

*Proof.* Let $\{C_{\sigma_i}\}$ be the set of cycles of the form $C_{\sigma_i} = (x_0x_1\ldots x_k)$ generated by an ordering on cycle set $\Gamma$. By Corollary 2.68, we have a correspondence between the edges $e \in E(\cup_i C_{\sigma_i})$ and the set $[n]^k$. By the definition of stable ordering, the graph $\cup_i C_{\sigma_i}$ is connected. Thus by Lemma 2.85, $\cup_i C_{\sigma_i}$ is Eulerian. Therefore there exists a universal cycle of $k$-subsets of $[n]$. \qed
3 Worked Examples

3.1 [9, 2]

Example 3.1. For \( k = 2 \) and \( n = 9 \), we have a cycle set with a stable ordering.

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<tr>
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We have universal cycle:

\[ 146258279357368478349569245167189123. \]
3.2 \([10, 3]\)

**Example 3.1.** For \(k = 3, n = 10\), we have a cycle set and stable ordering.

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![T Diagram](image)

We have universal cycle: 10 267856179612910 6814710 13510 23 4591456734128712569357245810 1468237924810 3789235610 457 910 346710 246925834910 57138910 158913610 123689478.
3.3 \([11, 4]\)

**Example 3.3.** Let \(k = 4, n = 11\). We have a cycle set with stable ordering.

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<th>Cycle</th>
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<td>{1,3,6,9,11}</td>
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<tr>
<td>{1,3,7,8,10}</td>
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<td>{1,4,5,6,7}</td>
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<td>{5,6,8,10,11}</td>
<td>(56 11 8 10)</td>
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<tr>
<td>{1,4,5,8,10}</td>
<td>(1458 10)</td>
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<td>{5,7,8,9,10}</td>
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We have universal cycle:

\[
14368 \ 11 \ 7169 \ 10 \ 7168 \ 11 \ 235791358 \ 11 \ 1369 \ 11 \ 1356 \ 10 \ 1378 \ 10 \ 1
357 \ 11 \ 28 \ 10 \ 7 \ 11 \ 2367 \ 11 \ 10 \ 359 \ 11 \ 10 \ 36712358 \ 10 \ 2356923 \ 10 \ 7
9 \ 11 \ 4 \ 10 \ 792354123 \ 10 \ 11 \ 1258 \ 11 \ 925871259 \ 10 \ 1268 \ 10 \ 125
6 \ 11 \ 1279 \ 11 \ 12384723891247 \ 10 \ 124896 \ 11 \ 4857 \ 11 \ 10 \ 157 \ 11 \ 48
9 \ 10 \ 5726 \ 10 \ 5789 \ 10 \ 248 \ 11 \ 12467 \ 11 \ 245671459 \ 11 \ 146 \ 10 \ 11 \ 9
2678914786 \ 10 \ 4789 \ 11 \ 10 \ 189 \ 11 \ 378926 \ 10 \ 11 \ 1458 \ 10 \ 145682457 \ 10 \ 3
48 \ 11 \ 10 \ 3456918569 \ 10 \ 456 \ 11 \ 8 \ 10 \ 56 \ 11 \ 7956 \ 11 \ 345893467934
579245 \ 10 \ 11 \ 2439 \ 11 \ 2463 \ 10 \ 2469123689 \ 10 \ 368573681439 \ 10 \ 1437 \ 11
\]

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Appendices

The purpose of Appendix A is to provide sufficient background information so that a mathematically mature reader may appreciate and understand the content of my thesis. The topics discussed are hardly complete and further reading is recommended. In Appendix B, I provide my perspective on the significance of mathematics research.

A Background Material

A.1 Graph Theory

A graph $G$ is an ordered pair $(V, E)$ where $V = V(G)$ is the vertex set of $G$, and $E = E(G)$ is the edge set of $G$. $E(G)$ consists of unordered pairs of elements from $V(G)$. Given $u, v \in V(G)$, the edge $e = (u, v)$ depicts a relation between vertices $u$ and $v$ and is represented as a line connecting $u$ to $v$. We call $u$ and $v$ the endpoints of edge $e$. Note that we often write edge $(u, v)$ as $uv$.

**Example A.1.** Graph $L$ has vertex set $V(L) = \{a, b, c, d, e\}$ and edge set $E(L) = \{ab, ad, ae, be, bd, cd\}$.

![Graph L](image)

Edge $ab$ has endpoints $a$ and $b$. Notice that vertex $a$ is the endpoint of three edges in $E(L)$. We say that vertex $a$ has degree three in graph $L$. We write $d_L(a) = 3$. If it is clear that we are discussing graph $L$, then we can abbreviate this notation to $d(v) = 3$.

For $u, v \in V(G)$, a $uv$-walk $W$ is a sequence of vertices $u = v_0, v_1, \ldots, v_{k-1}, v_k = v$ such that for $1 \leq i < k$ there is an edge $(v_i, v_{i+1}) \in E(G)$. A graph $G$ is connected if it contains a $uv$-walk for any $u, v \in V(G)$.

In our graph $L$, the sequence $e, a, d, b, e, a, b, d, c$ provides an $ec$-walk. Since we can find a $uv$-walk in $L$ for every $u, v \in V(L)$, we know that $L$
is connected.

If a graph $G$ contains a $uv$-walk for all $u, v \in V(G)$ such that $d(u) > 0$ and $d(v) > 0$, then we say that $G$ has at most one nontrivial component.

**Example A.2.** Consider graph $M$ and graph $N$.

Graph $M$ has one nontrivial component because the only vertex that is disconnected from the other vertices in $V(M)$ is $d$ and $d_M(d) = 0$. However, $M$ is disconnected since there is no $cd$-walk in $M$. Similarly, graph $N$ is disconnected since there is no $cd$-walk in $N$. We know that $N$ has more than one nontrivial component because there are no vertices in $N$ of zero degree.

A *circuit* $C$ is a $uv$-walk such that $u = v$ and an edge $e$ appears in $C$ at most one time. An *Eulerian circuit* in a graph $G$ is a circuit that contains every edge in $E(G)$. We say that a graph $G$ is *Eulerian* if $G$ contains an Eulerian circuit.

**Theorem A.3.** A graph $G$ is Eulerian if and only if it has at most one nontrivial component and its vertices all have even degree.

For a proof, consult Douglas West’s “Introduction to Graph Theory”[9].

A *directed graph* $D$ is an ordered pair $(V, E)$ such that the edges in $E(D)$ are ordered pairs of elements from $V(D)$. We draw edges in a directed graph as an arrow. If $uv$ is an edge in $E(D)$ then $u$ is the tail of the edge and $v$ is the head. We say that $uv$ is directed from $u$ into $v$.

**Example A.4.** Directed graph $Z$ has vertex set $V(Z) = \{a, b, c, d, e\}$ and edge set $E(Z) = \{ab, ad, ae, ba, be, ca, cb, db, dc, ec, ed\}$. Notice that $ab$ and $ba$ are distinct edges.
Edge $ab$ has $a$ as its tail and $b$ as its head. Notice that vertex $a$ is the tail of three edges and the head of two edges in $E(Z)$. We say that vertex $a$ has out-degree three and in-degree two in graph $Z$. We write $d^+_Z(a) = 3$ and $d^-_Z = 2$. If it is clear that we are discussing graph $Z$, then we can abbreviate this notation to $d^+(a), d^-(a)$.

If we remove the ordering on the edges of a directed graph $D$ we have the underlying graph $G$.

**Example A.5.** Consider $Z$ from our previous example. The graph below is the underlying graph of $Z$.

![Graph](image)

### A.2 Set Theory

Roughly speaking, a set is a collection of objects called elements. If $x_1, x_2, \ldots, x_k$ are all the members of a set, then we express $S$ as $\{x_1, x_2, \ldots, x_k\}$. If $S$ is a set and $x$ is an element of $S$ then we write $x \in S$. The empty set contains no elements and is denoted $\emptyset$.

Let $S$ and $T$ be two sets. A subset $T$ of $S$ is a set such that if $t \in T$, then $t \in S$. We write $T \subseteq S$. Two sets $S$ and $T$ are equal, written $S = T$, if they have the same elements.

**Example A.6.** Consider sets $S$ and $T$.

![Sets](image)
We can write \( S = \{1, 2, 5, 6\} \) and \( T = \{1, 2, 7\} \). The set \( \{1, 5\} \) is a subset of \( S \) because \( 1 \in S \) and \( 5 \in S \). We know that \( S \neq T \) because \( 6 \in S \) and \( 6 \notin T \).

The union of two sets, denoted \( S \cup T \), is the set of all elements \( x \) such that \( x \) is an element of \( S \) or \( x \) is an element of \( T \). We write this in set notation as \( S \cup T = \{x : x \in S \text{ or } x \in T\} \). The intersection of two sets, denoted \( S \cap T \), is all elements \( x \) such that \( x \) is an element of \( S \) and \( x \) is an element of \( T \). This is written in set notation as \( S \cap T = \{x : x \in S \text{ and } x \in T\} \). The set difference \( S \setminus T \) contains the elements of \( S \) that are not in \( T \). I.e., \( S \setminus T = \{x : x \in S \text{ and } x \notin T\} \).

**Example A.7.** Continue with \( S \) and \( T \) from our previous example.

Then \( S \cup T = \{1, 2, 5, 6, 7\} \) and \( S \cap T = \{1, 2\} \). We can take the subset \( \{1, 5\} \) of \( S \) and \( S \setminus \{1, 5\} = \{2, 6\} \). Then \( (S \setminus \{1, 5\}) \cap T = \{2\} \).

The power set of a set \( S \), denoted \( \mathcal{P}(S) \), is the set of all subsets of \( S \), including \( S \) and \( \emptyset \).

**Example A.8.** Let \( T = \{1, 2, 7\} \). Then

\[
\mathcal{P}(T) = \{\emptyset, \{1\}, \{2\}, \{7\}, \{1, 2\}, \{1, 7\}, \{2, 7\}, \{1, 2, 7\}\}.
\]

### A.3 Combinatorics

One of the main goals of combinatorics is the enumeration of sets of elements. We will review several quantities that are recurring throughout the study of \( k \)-subsets of \( n \).

Given a finite set \( S \). The cardinality or size of \( S \), denoted \( |S| \), is the number of elements in \( S \).
Example A.9. The set $S$ from Example A.6 has four elements. Thus $|S| = 4$.

Given a collection of sets, the principle of inclusion-exclusion allows us to evaluate the size of their union. This technique is particularly useful when we can’t explicitly determine the elements in a union of sets.

![Venn Diagram](image)

Let’s consider a simple example of two sets $A$ and $B$. We want to know the size of their union, i.e. $|A \cup B|$. If we add $|A| + |B|$ we can get a sum that is greater than $|A \cup B|$ because $A$ and $B$ may have elements in common. However, if we subtract $|A \cap B|$ from $|A| + |B|$ we eliminate the overcount. In other words, $|A \cup B| = |A| + |B| - |A \cap B|$.

Example A.10. For $A = \{a, b, c\}$ and $B = \{a, c, d, e\}$, let’s use the inclusion-exclusion principle to evaluate $|A \cup B|$. We have $|A| = 3$ and $|B| = 4$. Notice that $A$ and $B$ have the elements $a$ and $c$ in common. So $|A \cap B| = 2$. Therefore we should have $|A \cup B| = 3 + 4 - 2 = 5$. Indeed $A \cup B = \{a, b, c, d, e\}$, a five element set.

Consider the word 1234. We can rearrange the digits in this word to form a new word such as 2341. We say we permuted the digits in the word. Formally, a permutation of $n$ distinct elements is a bijective mapping $f: \{1, 2, \ldots, n\} \mapsto \{1, 2, \ldots, n\}$. We can ask how many permutations exist of $n$ elements. I.e., how many ways can we order $n$ elements?

Lemma A.11. There are $n!$ permutations of $n$ elements.

Note that $n! = n(n-1)(n-2) \ldots (2)1$.

Proof. Consider a set of $n$ elements. We choose one of the $n$ elements to be in the first position of our word. We choose among the $n-1$ remaining elements to fill the second position. Then $n-3$ options remain to be chosen.
for the third position. We continue until 1 element remains to be placed in the $n$th position. To count the total number of options, multiply $n(n-1)(n-2)\ldots(2)1$.

Consider how many length two words with distinct digits we can make using the integers \{1, 2, 3, 4\}. There are 12 words, namely 12, 21, 13, 31, 14, 41, 23, 32, 42, 43, 24, and 34.

In general, we have the following lemma.

**Lemma A.12.** There are \(\frac{n!}{(n-r)!}\) ways to create length $r$ words with distinct digits from $n$ characters.

**Proof.** As in the proof of the previous lemma, we choose one of the $n$ elements to be in the first position of our word. We choose among the $n-1$ remaining elements to fill the second position. We continue until $n-r+1$ elements remain to be placed in the $r$th position. To count the total number of options, multiply $n(n-1)\ldots(n-r+1)$.

We frequently discuss the set \(\left[\begin{array}{c}n \\ k\end{array}\right]\) all $k$-subsets of the integers \{1, 2, \ldots, $n$\}. Consider the number of 2-subsets of \{1, 2, 3, 4\}. There are 6 of them: \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, and \{3, 4\}.

We can use the previous lemmas to count the number of $k$-subsets of any $n$-set.

**Lemma A.13.** \(\left[\begin{array}{c}n \\ k\end{array}\right]\) = \(\binom{n}{k}\).

Note that a *binomial coefficient*, written \(\binom{n}{k}\), is equal to \(\frac{n!}{(n-k)!k!}\).

**Proof.** Recall that there are \(\frac{n!}{(n-k)!}\) ways to create distinct length $k$ words from $n$ characters. Since sets are unordered, an element can appear in any position of the set. Thus we divide the total number of length $k$ words by the $k!$ permutations of a word.

We can generalize this to a *multinomial coefficient*, written \(\binom{n}{k_1,k_2,\ldots,k_j}\) for $k_1 + k_2 + \cdots + k_j = n$. One combinatorial interpretation of a multinomial coefficient is the number of ways of depositing $n$ distinct objects into $j$ distinct bins, with $k_1$ objects in bin 1, $k_2$ objects in bin 2, \ldots, and $k_j$ objects in bin $j$.

Thus, \(\binom{n}{k_1,k_2,\ldots,k_j} = \binom{n}{k_1} \binom{n-k_1}{k_2} \binom{n-k_1-k_2}{k_3} \cdots \binom{n-k_1-k_2-\cdots-k_{j-1}}{k_j}\). To see this, we have \(\binom{n}{k_1}\) ways to select $k_1$ objects for bin 1. There are $n-k_1$ objects remaining to be
chosen for bin 2. In general, there are \( n - (k_1 + \cdots + k_{l-1}) \) objects remaining to be chosen for bin \( l \). At bin \( j \) we have \( n - (k_1 + \cdots + k_{j-1}) = k_j \) objects remaining to be chosen.

We can simplify this expression to 
\[
\binom{n_{k_1,k_2,\ldots,k_j}}{n} = \frac{n!}{k_1!k_2!\ldots k_j!}.
\]
B Significance of the Research Process

Application plays a key role in the general public’s perception of mathematics. This is unsurprising. Throughout the past century, the United States government has pushed math education in this direction. In fact, my first research experience was a National Science Foundation (NSF) funded program at East Tennessee State University. According to the NSF Act of 1950, this organization’s purpose is “to promote the progress of science; to advance the national health, prosperity, and welfare; and to secure the national defense.”

While these goals are admirable, in no way can I relate them to my work. I entertain questions about application because I understand the greater context and background that frames mathematical perception and generates this type of question. However, I have no answer to the question “what is this good for?” At least, I have no answer concerning application.

After fifteen years of being taught math through applications to physics, economics, and even daily life, I was appalled to learn that much of pure mathematics is “good for nothing”. At Wellesley I began to reassess mathematics. Even though the material I learned can’t cure a disease or prevent an economic recession, I felt that I was studying something worthwhile. I had to reconcile this sense of purpose with the pervasive notion that math needs an application.

Interestingly, it was not in the math department, but in the art and philosophy departments that I began formulating an answer to this puzzle. Late one night in fall 2009 I was working in one of the Pendleton Hall drawing studios. Because I had a linear algebra exam to take in the morning and a drawing study due the following day, I had a chalkboard on my right and an easel on my left. Every time I needed to take a break from the charcoal, I would turn to the board, pick up the chalk, and work on a proof. Although I thought nothing of this at the time, the memory inspired a comparison of mathematics to art production.

While reflecting on my practice as an artist, it was clear that the activity of generating math with chalk on slate and drawing with charcoal on paper are quite similar. More importantly, I noticed that my mental processes as a mathematician and as an artist are analogous in striking and profound ways.
With this parallel between the activity of mathematics and art production in mind, I completed a course about philosophy of art that equipped me to pursue and articulate this analogy.

I believe that mathematics itself is a form of art and that the activity of mathematics research is a type of art production. As a consequence, when someone asks “What is this good for?” I feel no pressure to generate a compelling response. Mathematics is independently valuable, and we don’t need to look elsewhere to justify our practice.

The following is the philosophical explanation of how I reached this conclusion. I begin by examining the claim that mathematics is beautiful.

**Mathematical Beauty**

“Beauty in mathematics is seeing the truth without effort”
- George Pólya, Hungarian mathematician (1887-1985)

Mathematics, in the conventional sense, is the systematic treatment of numbers, magnitude, and form, and the relationships between these quantities expressed symbolically. The place of beauty and what constitutes it within these concepts is vague and widely debated. Mathematician and philosopher Gian-Carlo Rota remarks, “Theories that mathematicians consider to be beautiful seldom agree with the mathematics thought to be beautiful by the educated public” [8]. The educated public tends to ascribe aesthetic language to mathematics that they can visualize, such as classical Euclidean geometry, while professional mathematicians generally focus on the more abstract forms that they work with on a daily basis. Accordingly, we can consider mathematical beauty in two main categories: the visual and the abstract.

Many mathematical objects lend themselves to aesthetic appreciation in the same way that a painting or drawing is found beautiful. This is not surprising considering the long historical relationship between mathematics and art. Ancient Egyptians and Greeks used the Golden Ratio $\phi$, a geometric constant defined by $\phi^2 - \phi - 1 = 0$, within their pyramids and the Parthenon. Modern day use of fractals and tessellations in artwork reveals math’s continuing presence within the art world. It is hardly a step for viewers to abstract the underlying mathematics, the structure of the art object, and apply the same
aesthetic terms. Any portrayable mathematical object that does not require cognitive understanding of the underlying principles falls within the visual category of mathematical beauty. Objects of visual mathematical beauty include: Euclid’s five platonic solids, Pascal’s triangle, and the Fibonacci numbers.

The points of controversy with aesthetic classifications in mathematics generally come from mathematicians themselves. Professionals have found various theorems, definitions, and proofs beautiful at different points. Because aesthetic appreciation of these forms involves thorough familiarity with the mathematical concept, they are abstract. Examples of abstract mathematical beauty include: Fermat’s last theorem, Galois Theory of equations, and the Weierstrass approximation theorem.

Beautiful abstract constructions exhibit a wide range of characteristics. Elegance is a favored term, and it can be applied to a proof that is surprisingly succinct or one that doesn’t rely heavily on outside assumptions. English Mathematician G. H. Hardy, in his essay A Mathematician’s Apology, constructs a tool-kit for defining mathematical beauty, a collection of qualities such as significance and “unexpectedness, combined with inevitability and economy” that discern beautiful theorems or proofs [5]. Although many proofs share these qualities, only some are typically found beautiful. Perhaps more delineating are descriptions of “ugly” mathematics: clumsy, trivial, awkward, redundant, or ‘quick and dirty’. Rota emphasizes, “Lack of beauty is associated with lack of definitiveness” [8]. Definitiveness speaks to the structure and form of the mathematics, which leads to a key observation: there is a definitive structure and form inherent to all beautiful mathematics.

Mathematics can be beautiful. However, we must also consider, how is it possible that we can aesthetically appreciate entities that we are unable to experience with our senses.

Aesthetics in Cognition

“There is neither a science of the beautiful, only a critique, nor beautiful science, only beautiful art.”

Immanuel Kant, German Philosopher (1724-1804)

By definition, mathematics is a cognitive activity. Although uneducated
viewers can appreciate visual forms of mathematical beauty, abstract mathematical beauty demands a proficiency in the given area of mathematics. Yet, mathematicians claim to gain aesthetic pleasure from their work.

Beautiful art and beautiful mathematics share many of the same characteristics, but the perception of these qualities must be different. Art has a concrete physical presence while abstract mathematics by definition cannot posses the same materiality. Research Physicist Gideon Engler examines this relationship with regards to unity: “In art [unity] is as a ‘whole’ which is more than the parts and is not deducible from them, whereas in science unity is also a ‘whole’ and is also more than its parts, but it can be perceived only through the analysis of its parts. For example, Einstein’s famous mass-energy relation, $E = mc^2$, may be considered as such a unity, but each term is a symbol for concepts that can be understood in relation to other terms” [4].

With abstract mathematics, the aesthetic presence of an object comes from the cognitive parts, which operate as the medium. In the same way charcoal and paper are the means for drawing, knowledge of energy, mass, and the speed of light are the anatomy of $E = mc^2$. Without this understanding, the formula is a meaningless jumble of letters and the viewer cannot experience the unity and other aesthetic characteristics of the object.

The ability to perceive mathematical objects is vital to experiencing abstract mathematical beauty. Mathematician Andrew Wiles describes his mathematical experiences in terms of entering a dark mansion: “One goes into the first room, and it’s dark, completely dark. One stumbles around bumping into the furniture, and gradually, you learn where each piece of furniture is, and finally, after six months or so, you find the light switch. You turn it on, and suddenly, it’s all illuminated” [7].

Stumbling amongst the furniture is the cognitive process involved with abstract mathematical beauty. The viewer must become knowledgeable on each part of the object. At the point of illumination, the viewer has sufficient understanding to piece the parts into a whole. When the viewer turns to look at the mathematical object, they already cognitively know the form. Aesthetic judgment involves synthesis, while epistemic judgment involves separating the abstract entity into its constituent elements. In the same way a botanist
can know what a flower is and still claim a flower is beautiful, the mathematician can make an aesthetic judgment.

Many philosophers and mathematicians alike argue that no distinction between aesthetic and epistemic judgment exist. Gian-Carlo Rota reasons, “‘Mathematical Beauty’ is the term [mathematicians] have resorted to in order to obliquely admit the phenomenon of enlightenment . . . Mathematicians may say that a theorem is beautiful when they really mean to say that the theorem is enlightening” [8]. The moment of enlightenment can be compared to the instant Wiles described as illumination in his dark mansion. Enlightenment provides the overall sense of the mathematical object. Rota suggests that mathematical beauty is avoiding the admission of enlightenment. While undoubtedly some mathematicians misuse mathematical beauty to express their overall comprehension of a topic, Rota’s assessment is particularly harsh and ignores the validity of the aesthetic characteristics used to describe beautiful mathematics. One could aptly compare enlightenment to the ability to see an artwork. It is that moment when all of the cognitive elements slip into a cohesive whole, allowing the viewer to perceive the object. Enlightenment is just one more essential step towards the ability to make an aesthetic judgment. It does not invalidate the order, symmetry, unity, or elegance of an object, and the viewer’s ability to discern these characteristics.

Given that abstract mathematical entities can be beautiful, I examine where these mathematical objects originate.

Constructivism V. Realism
“*The Good Lord made all the integers; the rest is man’s doing.*”

Leopold Kronecker, German mathematician (1823-1891)

The history of mathematics has a rich tradition of mathematicians and philosophers speculating on the nature of mathematics and the way in which we have knowledge of mathematical entities. The Greek philosopher Plato was the first to form a coherent philosophy. Platonism, a notion from the middle dialogues, suggests that mathematical entities are abstract, eternal, and unchanging [1]. Platonism became a foundation for mathematical realism; the idea that math objects are independent, and all that mathematicians can do is discover their properties. Realism was the predominant view until very recently.
Several developments have cast doubt upon mathematical realism. In the early 1900’s Alfred North Whitehead and Bertrand Russell published a three-volume work *Principia Mathematica*, attempting to derive all mathematical truths from a well-defined set of axioms and pure logic. Although extremely important in mathematical logic and philosophy, the texts were not immaculate: the truthfulness of the basic theories upon which the work rested was inconclusive. In 1931, Austrian mathematician Kurt Gödel published his incompleteness theorems and destroyed the *Principia*’s attempt to prove the certainty of mathematics.

The inability to prove the foundation of mathematics led scholars to a new philosophy: constructivism. Constructivism is the idea that we construct or create the mathematical objects we discuss [1]. In the 1950’s Hungarian philosopher of mathematics Imre Lakatos challenged modern mathematical dogmatism. Lakatos further argued that the activity of mathematics is the essence of mathematics, challenging the idea that the activity of mathematicians is to simply enlarge our understanding of an unchanging body of knowledge [6]. The implications of this philosophy are far reaching. The creativity inherent in constructivism means mathematics is the product of human imagination. Meanwhile, Gian-Carlo Rota reasons that creativity is a vacuous term and that “beauty cannot be directly sought after”. He further argues, “Mathematicians work to solve problems and to invent theories that will shed light upon the world, not to produce beautiful theorems and pretty proofs” [8]. Within the old notion of realism, Rota may be correct. But the infallibility of realism has been disproven and mathematical philosophy is working under a new order.

Today, realism and constructivism are the primary contrasting ideologies, dividing the mathematical community into two main groups. Unfortunately, many aspects of the theories are diametrically opposed. For the realist there is no problem in admitting mathematical objects, whether or not they are well-defined. This is incoherent for the constructivist. Both philosophies have some supporting evidence and the correct answer feasibly lies between the two camps.

With this shift towards constructivism in mathematics, we need to re-evaluate the mathematician and the product of their labor.
Mathematician as Artist

“Mathematics, as much as music or any other art, is one of the means by which we rise to a complete self-consciousness. The significance of Mathematics resides precisely in the fact that it is an art; by informing us of the nature of our own minds it informs us of much that depends on our minds.”

- J. W. N. Sullivan, English science writer (1886-1937)

Mathematical objects, such as the natural numbers, do exist independently in nature. Due in part to the failings of set-theory-logic in proving all of mathematics, modern mathematics is no longer the quest for absolute truth. Mathematical beauty is a prevalent reason for practicing mathematicians to be drawn to their field, and many modern mathematical objects are created. Irish mathematician William Rowan Hamilton, for example, is known as the inventor of Quaternions, a type of higher complex number. While the professional field has shifted towards this focus, one philosophical implication has yet to be fully explored: when we find the creation of the mathematician beautiful, the mathematician becomes an artist and their work an art form. The research process is art production where abstract mathematical entities are substituted for a physical medium.

In the same way that not all human activity via physical material is art, not all math is art. But there are many pronounced cases where the line between mathematics and art is blurred to the point that mathematics is art. Hegel introduced the idea that art is a way in which humans come to recognize things about themselves. In the same way an artist thinks through the materiality of their own activity to the point of self-replication and self-realization, the mathematician mediates themselves through the mathematics. Andrew Wiles’ proof of Fermat’s Last Theorem itself is an embodiment of the mathematician’s toil, an incarnation of his mind within the object.

Mathematics can be a moment of intense feeling and original thought. In the same way a sketch-book helps a draughtsman to emerge into the light of consciousness, a proof becomes a composition, an embodiment of a mathematician’s reconciliation with the world. Mathematics begins as discovery, and a child’s first calculations on paper exemplify the desire to understand and to clarify. As a mathematician nears the unknown, however, his practice becomes vastly more complex and deeply personal. Not only is mathematics
the process of comprehension and discernment, the act becomes creative, an accurate recording of human thought and experience.

Mathematics can be a uniquely personal experience. It becomes an exploration, an expedition into self, and, perhaps, an honest chronicle of our time here on earth. At the conclusion of *The Abuse of Beauty*, philosopher Arthur C. Danto contends, “Beauty is an option for art and not a necessary condition. But it is not an option for life. It is a necessary condition for life as we would want to live it.”

Similarly, beauty is not a requirement of mathematical discovery. Many have stripped math down to practical application, merely a tool in daily life. Others imagine a mathematical utopia free from disappointments, ambiguities, and failure; a world where truth is absolute. I take issue with these notions. The charm of mathematics is not utility or truth, but its value to the human experience. Mathematics is a fundamental and profound product of human activity. Indeed, beauty is not necessary in mathematics. But it is not an option for the mathematician. It is necessary in the world where we would want to live.
References


