2012

Comaximal Ideal Graphs of Commutative Rings

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Comaximal Ideal Graphs of Commutative Rings

A THESIS SUBMITTED

TO THE DEPARTMENT OF MATHEMATICS

OF WELLESLEY COLLEGE

IN PARTIAL FULFILLMENT OF

THE PREREQUISITE FOR HONORS

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May 2012

Acknowledgments

I would like to begin by thanking my advisor, Professor Alexander Diesl, for his encouragement and excitement for my research. He has always remained supportive and interested in my work, and his many questions have kept me motivated and excited about my project. I would like to thank him as well for being understanding about my silly questions, helping me piece apart my convoluted arguments, and being forgiving of my persistent and curious typos. Without him, this research would not have been nearly as much fun.

I have to thank Professors Andrew Schultz, Megan Kerr, and Emily Buchholtz for being willing to read through my thesis and serve as my thesis committee. I appreciate the time you will spread processing my results and asking me questions at my defense.

I would like to thank the honors support group, especially Professor Schultz and Melinda Lanius, for sharing my excitement about my project and math in general. Thank you for making sure that the meetings were always fun, and encouraging any number of digressions. I also would like to thank members Alex Gendreau and Liz Ferme for being encouraging and putting up with all of the thesis talk. I am sure you will both do wonderfully on the honors exam. In fact, I would really like to thank the entire math department, including my professors, Melanie Chamberlain, and all of my study buddies who have helped me survive my time at Wellesley.

Lastly, I need to thank my friends and family who have made my last year at Wellesley so wonderful. Thank you to my mother and father for believing in me and pushing me to do my best. Thanks to my sister Iulia for late night distractions when I really needed a break. Thanks to the other seniors with theses for worrying with me and getting excited with me about all of the projects being completed. And thanks to all of the other friends who have always been happy to share a smile and be frivolous.

Contents

1 Abstract

A commutative ring R can be represented as a graph whose vertices are the ideals of R, and in which two vertices v_1, v_2 are adjacent if and only if v_1 and v_2 are comaximal. This graph, denoted $\widetilde{\mathcal{G}}(R)$, is called the comaximal ideal graph of R and is a variant on the graphical ring representations of Beck and of Sharma and Bhatwadekar [4, 11]. Because the properties of $\widetilde{\mathcal{G}}(R)$ are derived from the lattice of ideals, I have been able to use this ring representation to highlight overall structural properties not visible with the other element based representations. Using lattice theory I have shown that there are specific relations with the clique number, chromatic number and number of partitions of a graph, and that there is a correspondence between the vertices not contained in the Jacobson radical $J(R)$ and those contained in $J(R)$.

2 Introduction

Mathematicians sometimes approach problems by looking at the big picture rather than focusing on smaller details. Understanding the overall structure of a problem or a mathematical object can highlight the importance of the certain details. To this end, I have focused my work on studying the ideal structure of commutative rings with identity, rather than focusing on the actual elements and how they interact. Because ideals are sets of ring elements, limiting research to ideals can illustrate properties that rings have in common. The set of ideals is usually smaller than the set of elements, so considering ideals is less complicated. Looking at just the ideals can show how similar or dissimilar rings are with each other. To study the ideal structure of commutative rings I used graph theory and lattice theory.

In 1988, Istvan Beck proposed the study of commutative rings by representing them as graphs [4]. He defined the zero divisor graph for a ring to be the graph consisting of a vertex for each element in the ring, and an edge between two vertices v and w if vw is equal to zero. These zero divisor graphs marked the beginning of an approach to studying commutative rings with graphs. A basic question about this ring representation is, what graphs can represent rings? Attempts to answer this question involve looking at graph properties such as the chromatic number and maximal clique size to find rules about possible graph structures. Beck's work showed that the chromatic number of a zero divisor ring graph equals the clique number under certain easily satisfied conditions. Anderson and Naseer later showed that sometimes the clique number does not equal the chromatic number [2]. In the past seventeen years, researchers have sought to determine which algebraic properties were reflected in this graph or variations of it.

In 1994, P.K. Sharma and S.M. Bhatwadekar proposed a new approach that constructed another graph for the commutative ring R: the vertices are still elements of the ring, and there is an edge between two vertices a and b in R if $Ra + Rb = R$ [11]. I will refer to this graph as the *standard comaximal graph*. Once more, the authors focused on the question of which graphs were realizable as the standard comaximal graph of a ring. In contrast to Beck's graph, the properties of this graphical form relate more to the ideal structure of the ring rather than to the zero divisors. Their original use of this graphical form was to show that a ring is finite if and only if the corresponding graphical representation contains no infinite clique. Maimani et al. followed up on these preliminary results by analyzing properties of the diameter and connectedness of these graphs [9].

Although the definition of Sharma and Bhatwadekar's graph concerns the elements of the ring, the graph form really reflects the ideal structure rather than the elements. So it is natural to consider a graph which directly reflects the ideal structure. At the suggestion of my advisor, Professor Diesl, I chose to extend the research in [11, 9] by considering the graph where there is a vertex for every ideal of the ring, and two vertices are adjacent if and only if the corresponding ideals are comaximal. I explored how many of the properties from Maimani's work extend to this new graphical type, and I showed that many of these properties were actually intrinsically related to the lattice of ideals for the ring.

Note that throughout this paper all rings are considered to be commutative rings with unity unless otherwise stated. For access to basic definitions and theorems please reference the appendix at the end.

3 Three graphs used to represent rings

To begin we will properly introduce the three graphical constructions thus far.

The zero divisor graph: The vertices of this graph are defined to be the elements of R. The set of vertices $\{v_1, v_2\}$ is edge if and only if $v_1 v_2 = 0$.

The standard comaximal graph: The vertices of the graph are elements of R. The set $\{v_1, v_2\}$ is an edge if and only if $Rv_1 + Rv_2 = R$. This graph is denoted $\Gamma(R)$ by [9]. The subgraph $\Gamma_1(R)$ is the subgraph of $\Gamma(R)$ induced by the units of R and the subgraph $\Gamma_2(R)$ is the subgraph induced by non-unit elements.

Comaximal ideal graph: This graph was proposed by Professor Alexander Diesl. The vertices are the ideals of R. In this graph $\{v_1, v_2\}$ is edge if and only if v_1 and v_2 are comaximal. We will denote the comaximal ideal graph of R by $\widetilde{\mathcal{G}}(R)$ and we will let $\mathcal{G}(R)$ denote the subgraph of the comaximal ideal graph induced by the set of vertices which represent proper ideals. In general we look at the latter graph. Because it has far fewer vertices and edges, it is simpler while still retaining much of the information of the standard comaximal graph–especially for rings with a large number of ideals. Given $\mathcal{G}(R)$, we can construct $\widetilde{\mathcal{G}}(R)$ by adding a vertex for R, and adding edges between R and every vertex in $\mathcal{G}(R)$. So this graph is less redundant. When there is only one ring under consideration we will use $\mathcal{G}$ instead of $\mathcal{G}(R)$.

When there is no risk of confusion we will let elements such as M or I in a graph describe both the vertices of the graph and the ideals they are meant to represent.

Definition 3.0.1. *Analogous to the discussion of* $\Gamma_2(R) \smallsetminus J(R)$ *in [9], I will define* $\mathcal{J}(R)$ *to be the set of vertices which represent ideals contained in the Jacobson radical of R, and I will let* $\mathcal{G}_2(R)$ *be the induced subgraph* $\mathcal{G}(R) \smallsetminus \mathcal{J}(R)$ *of* $\mathcal{G}(R)$.

Lemma 3.0.2. *The set of vertices in* $\mathcal{J}$ *is precisely the set of vertices of degree zero.*

Proof. We will show that the degree of an ideal in the Jacobson radical is zero by contradiction. If $I_1 \in \mathcal{J}$ is adjacent to a vertex I_2 then I_1 is adjacent with the maximal ideal M containing I_2. But the Jacobson radical of a ring is the intersection of all of the maximal ideals, so $I_1 \subseteq M$ implies $I_1 + I_2 \subseteq M$. Hence we have a contradiction.

Suppose we have an ideal I of degree zero. Then this element is adjacent with no maximal ideal, which means that it is contained in every maximal ideal, so $I \subseteq J(R)$. $\square$

Based on Lemma 3.0.2, we understand that $\mathcal{G}_2(R)$ is the subgraph of $\mathcal{G}(R)$ induced by vertices with a positive number of adjacencies. So the chromatic number and clique number of both of these graphs is the same. Moreover, in many cases the number of vertices in $\mathcal{J}(R)$ can often be deduced from $\mathcal{G}_2(R)$, as can be seen in the proof of Theorem 7.1.3.

4 Properties of the construction for semilocal rings

A commutative ring is semilocal if it has finitely many maximal ideals. A ring is Artin if for any sequence of ideals I_1, I_2,, I_3... such that $I_i \supseteq I_{i+1}$ for all $i \in \mathbb{N}$, there is n such that I_n=I_{n+1}=....All Artin rings are semilocal, so the following results apply to all Artin rings.

4.1 Semilocal rings

The following results for semilocal rings relate the number of ideals with the paroning of $\mathcal{G}$ and with the chromatic number and clique number of $\mathcal{G}$. Note that the chromatic number, clique number, and number of partitions for $\mathcal{G}$ are the same as those for $\mathcal{G}_2$, since this is just the subgraph of $\mathcal{G}$ induced by vertices with positive degree.

Definition 4.1.1. *An n-partite graph is a graph which can have its vertices partitioned into n sets* $\mathcal{V}_1, ..., \mathcal{V}_n$ *such that, for each* $i \in \{1, ..., n\}$, *if* $v, w \in \mathcal{V}_i$ *then v and w are not adjacent.*

From basic graph theory we know a graph G is n-partite if and only if $\chi(G) = n$ [5]. Furthermore, the clique number of a graph is less than or equal to the chromatic number. So if G is n-partite then $clique(G) \leq n$. Therefore, if the clique number of G is greater than or equal to n then G is n-partite [5]. So the clique number must be at least as large as the number of maximal ideals, because they form a clique.

Theorem 4.1.2. *The ring R has n maximal ideals,* $M_1, ..., M_n$, *if and only if* $\mathcal{G}$ *is n-partite.*

Proof. Suppose that R is a semilocal ring with n maximal ideals, $M_1, ..., M_n$. By definition, any two distinct maximal ideals are comaximal, which means that the set of maximal ideals forms a clique of size n. So we know that the graph is at least n-partite. We will construct n partitioning sets, $\mathcal{V}_1$,...$\mathcal{V}_n$, as follows: begin placing M_i in $\mathcal{V}_i$ for i=1,...n. Recall that every ideal $I \neq R$ is contained in some maximal ideal M_i. (We know that any ideal may actually be contained in more than one maximal ideal). Choose M_i containing I, say the maximal ideal with the smallest subscript, and put I in $\mathcal{V}_i$. Do this for all of the ideals of R. So every ideal I in $\mathcal{V}_i$ is contained in M_i. Suppose I_1, I_2 are both in the arbitrary partitioning set $\mathcal{V}_i$. Then we have $I_1 + I_2 \subseteq M_i$, so I_1 and I_2 are not adjacent. So there are no edges connecting elements within any partitioning set, which means that the graph is n-partite.

Suppose that $\mathcal{G}$ is n-partite. We know that R is semilocal by contradiction. Indeed, if R has infinitely many maximal ideals, then each maximal ideal would need its own partitioning set. So the ring is semilocal with m maximal ideals. But by the forward direction, this means $\mathcal{G}(R)$ is an m-partite graph, but not $(m-1)$-partite, so we must have $n = m$. □

Corollary 4.1.3. *If a ring is semilocal with n maximal ideals then clique*$(\mathcal{G}) = n$

Proof. We know $\mathcal{G}$ is n-partite by Theorem 4.1.2. Because $\mathcal{G}$ is n-partite we know $clique(\mathcal{G}) \leq n$. But the subgraph of $\mathcal{G}$ induced by the set of maximal ideal vertices is a complete subgraph of size n. Hence $clique(\mathcal{G}) \geq n$. □

Proposition 4.1.4. *If clique($\mathcal{G}$)=n then R is a semilocal ring with n maximal ideals.*

Proof. By Corollary 4.1.3, if there were more than n maximal ideals, then $\mathcal{G}$ would contain a larger clique. So there must be at most n maximal ideals. If there were fewer than n maximal ideals, say m many in total, then by Theorem 4.1.2, the graph would be m-partite. By an earlier statement this means $clique(G) \leq m < n$, so we have a contradiction. Hence the ring has n maximal ideals. □

Corollary 4.1.5. *The following are equivalent:*

1. *A ring R has n maximal ideals.*
2. *$\mathcal{G}(R)$ is n-partite.*
3. $\chi(\mathcal{G}(R)) = n$
4. $clique(\mathcal{G}(R)) = n$

Proof. This follows from Theorem 4.1.2, Proposition 4.1.4, and Corollary 4.1.3, as well as basic graph theory. □

Recall that to form n vertex-partitioning sets there was choice involved for ideals contained in multiple maximal ideals. So this paroning is not unique for any ring in which a non-maximal ideal is contained in more than one distinct maximal ideal. Because there are multiple partitions of such rings, we can consider how the ideals relate within each partitioning set.

Lemma 4.1.6. *For two ideals I and J in R, if $I \subseteq J$ then $deg(I) \leq deg(J)$.*

Proof. For two ideals I and J in R, suppose $I \subseteq J$. For arbitrary ideal A in R, if $I + A = R$ $J + A = R$. So $deg(I) \leq deg(J)$. $\square$

Definition 4.1.7. *Two vertices in a graph are said to be equivalent if they are adjacent to the same set of vertices. Note that this means that the vertices are not adjacent with each other in $\mathcal{G}$, because no vertex is adjacent to itself.*

Proposition 4.1.8. *Let R be a ring. Suppose $\mathcal{G}$ is n-partite. Let $\mathcal{V}$ be a partitioning set under an arbitrary partition of $\mathcal{G}$ and let M be the maximal ideal in $\mathcal{V}$. Then M must be one of the vertices of highest degree in $\mathcal{V}$, and all of the ideals of highest degree are adjacent to the same vertices. So maximal ideals may be identified up to equivalence of adjacencies within each partitioning set.*

Proof. If $I \in \mathcal{V}$ then $I \subseteq M$, otherwise $I + M = R$ and $\mathcal{V}$ contains an edge. So by Lemma 4.1.6, then $deg(I) \leq deg(M)$. As in Lemma 4.1.6, if $I + J = R$ then $J + M = R$ since $I \subseteq M$. So M is adjacent to any ideal J which is adjacent to $I \in \mathcal{V}$. So M must be one of the vertices of highest degree. If there is another vertex $N \in \mathcal{V}$ with the same degree as M then it must be adjacent to the same vertices, because M is adjacent to all of the vertices that N is adjacent with. Hence all of the vertices with highest degree are adjacent to the same vertices. $\square$

Because we can have ideals which are indistinguishable by the graph $\mathcal{G}(R)$, and which can be interchanged under isomorphism, it seems natural to define the following equivalence quotient graph.

Definition 4.1.9. *Consider the equivalence classes for the equivalence relation in Definition 4.1.7. Define the equivalence quotient graph $Q(G)$ for graph G is to have these equivalence classes as vertices, and edges between two vertices if representatives from each equivalence class are adjacent. Note that this graph is obviously well-defined.*

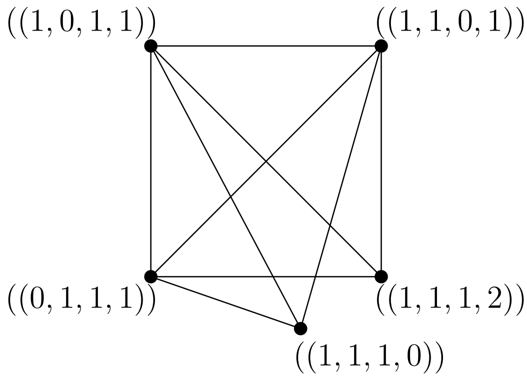
This graph may be used to relate the standard commutative graph with the comaximal ideal graph. For instance, let Γ be the standard comaximal graph for a ring. Then the set of vertices of $Q(\Gamma)$ is a subset of the set of principal ideals.

We can also define a second quotient graph of the standard comaximal graph $\Gamma(R)$, denoted $\check{Q}(\Gamma)$, where vertices of the original graph that generate the same ideal are identified. Note that this may not be as simple as identifying edges which share the same vertices. For example in $\mathbb{Z}_{12}$ the vertices 2 and 4 are identified, but they generate different ideals. If Γ has a complete subgraph, then $\check{Q}(\Gamma)$ will have a corresponding complete subgraph of the same size. This is true because two points that are adjacent to each other cannot represent the same ideal. In $\check{Q}(\Gamma)$ each ideal represented corresponds with an ideal in the comaximal ideal graph, so the complete subgraph can be found in the comaximal ideal graph representation too.

Recall that the motivation for the above results was the fact that a partition of the vertices of $\mathcal{G}$ may not be unique, as was shown in the proof of Theorem 4.1.2. Just as the partition may not be unique, a clique of size n need not be unique. The following example describes a ring where this is the case.

Example 4.1.10. *There are two possible cliques of size four in $R = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$.*

The graph $\mathcal{G}_2(R)$ contains a subgraph graph of the form:



where the subgraph induced by the set of vertices $\{((1, 1, 1, 2)), ((0, 1, 1, 1)), ((1, 0, 1, 1)), ((1, 1, 0, 1))\}$ forms a complete subgraph on four vertices. But the subgraph induced by the vertices $(1, 1, 1, 0)$, $((0, 1, 1, 1))$, $((1, 0, 1, 1))$, and $((1, 1, 0, 1))$ is a second distinct complete subgraph with four vertices. In fact the graph of any ring of the form $K_1 \times K_2 \times K_3 \times R$, where each of the K_i are fields and R is a local ring with three ideals, would contain multiple complete subgraphs with four vertices. This fact is because any field has only two ideals, (0) and (1). All graphs for rings of the form $K_1 \times K_2 \times K_3 \times R$, as above, would be isomorphic. We will see this latter in Proposition 6.1.3. In the following section we will further characterize types of rings which have cliques that contain non-maximal ideals. These are rings which are isomorphic to the finite direct product of local rings, where at least one of the rings is not a field.

4.2 Rings which are isomorphic to a finite direct product of local rings

Definition 4.2.1. *A ring is Artin if for any sequence of ideals I_1, I_2,, I_3... such that $I_i \supseteq I_{i+1}$ for all $i \in \mathbb{N}$, there is an integer n such that I_n=I_{n+1}=....*

This section will introduce notation used throughout this section discusses graphs of rings that are isomorphic to the direct product of finitely many local rings. Rings of this form are particularly important because, if R is an Artin ring, then $R \cong R_1 \times ... \times R_n$, where each of the R_i is a local Artin ring [3]. Moreover all finite rings are Artin rings.

In general we will let $R \cong R_1 \times ... \times R_n$ where R_i is a local ring. So without loss of generality we can write $R = R_1 \times ... \times R_n$. We define M_i to be the maximal ideal of R_i, for $i \in \{1, ..., n\}$. Then the maximal ideals of R are $N_i = R_1 \times ... \times R_{i-1} \times M_i \times R_{i+1} \times ... \times R_n$. All of the ideals of R are of the form $I_1 \times ... \times I_n$, where each I_i, is an ideal of the ring R_i. I will speak of I_i as **the ideal contribution of ring** R_i. Two ideals $I_1 \times ... \times I_n$ and

$J_1 \times ... \times J_n$ are comaximal if and only if I_i and J_i are comaximal for all $i \in \{1, ..., n\}$. This last fact is actually true even if any number of the R_i are not local. For rings which are the direct product of local rings, the set of all elements that can be in a maximal clique is precisely the set of elements of the form $I = R_1 \times ... \times R_{i-1} \times I_i \times R_{i+1} \times ... \times R_n$, where I_i is strictly contained in R_i and $i \in \{1, ..., n\}$.

Theorem 4.2.2. *For a ring $R = R_1 \times ... \times R_n$, the direct product of local rings, the ring R_i is a field if and only if N_i is the only element in the partitioning set $\mathcal{V}_i$ which can be part of a clique of size n.*

Proof. A clique of size n consists precisely of ideals of the form $R_1 \times ... \times R_{i-1} \times I_i \times R_{i+1} \times ... \times R_n$, one for each i. Indeed, for each i, if I is such an ideal in the partitioning set $\mathcal{V}_i$, then I must be comaximal with some element $I_j \in \mathcal{V}_j$ for each $j \neq i \in \{1, ..., n\}$. By Lemma 4.1.6 we know I must be comaximal with N_j, since $I \subseteq N_i$. So because each R_j is local, the ring contribution from R_j must be R_j itself.

Now suppose, on the other hand, that R_i is a field, which is true if and only if the only ideals in R_i are (0) and (1). Then the only element in $\mathcal{V}_i$ which can be part of a clique of size n is $R_1 \times ... \times R_{i-1} \times (0) \times R_{i+1} \times ... \times R_n$. Suppose that the partitioning set $\mathcal{V}_i$ contains only one element which can be part of a clique of size n. We know $\mathcal{V}_i$ must contain $R_1 \times ... \times R_{i-1} \times I \times R_{i+1} \times ... \times R_n$ for any ideal $I \subset R_i$, and that (0) is one such I. So the only ideals in R_i are (0) and (1), which means that R_i is a field. $\square$

Corollary 4.2.3. *Let $R = R_1 \times ... \times R_n$ equal the direct product of n local rings. Then $\mathcal{G}$ has a unique clique of size n if and only if R_i is a field for all $i \in \{1, ..., n\}$.*

Proof. If the clique of size n is unique, then each set in each partitioning set $\mathcal{V}_i$ can only contribute one element to the clique. For each i we know N_i is always such an element, so by Theorem 4.2.2 we know R_i is a field. If each R_i is a field, then by Theorem 4.2.2 we know that each partitioning set can only contribute N_i to the ring product, so the clique is unique. $\square$

The lack of a unique clique of size n motivates the introduction of the following subgraph.

Definition 4.2.4. *We define the subgraph $\mathcal{G}'$ of $\mathcal{G}_2$ to be the graph induced by the set of vertices of the form $R_1 \times ... \times R_{i-1} \times I_i \times R_{i+1} \times ... \times R_n$. The vertices of this graph are precisely vertices which can be part of a complete subgraph of size n.*

Note that if R_i contains a_i ideals including itself, then by the discussion in the proof of Theorem 4.2.2 the partitioning set $\mathcal{V}_i$ has exactly $a_i - 1$ vertices which could be part of a clique of size n. Indeed, elements from $\mathcal{V}_i$ which can be part of a clique of size n are of the form $R_1 \times ... \times R_{i-1} \times I \times R_{i+1} \times ... \times R_n$ with $I \in \mathcal{V}_i$., and there are only $a_i - 1$ possibilities for I. So each $\mathcal{V}_i$ contains $a_i - 1$ many vertices. Hence the number of vertices in $\mathcal{G}(R)'$ is $\sum_1^n (a_i - 1)$. Also, the number of cliques of size n is precisely $\prod_{i=1}^n (a_i - 1)$, since any collection of n elements of the form $R_1 \times ... \times R_{i-1} \times I \times R_{i+1} \times ... \times R_n$, with one element for each $\mathcal{V}_i$, will induce a complete subgraph with n vertices.

4.3 A special case, rings of the form $\mathbb{Z}_{n_1} \times ... \times \mathbb{Z}_{n_r}$

If a ring is of the form $\mathbb{Z}_{n_1} \times ... \times \mathbb{Z}_{n_r}$, maximal ideals are of the form $((1, ..., 1, p_i, 1, ..., 1))$, where p_i is a prime factor of n_i. Because $\mathbb{Z}_{n_i}$ is a principal ideal ring for all $n_i \in \mathbb{Z}^+$, two ideals $(I_1, ..., I_r)$ and $(J_1, ..., J_r)$ are comaximal if and only if I_i and J_i are coprime for all i between 1 and r. In fact, this case generalizes to all Artin principle ideal rings. Recall that all Artin rings are isomorphic to a unique (up to isomorphism) finite direct product of Artin local rings. If an Artin ring $A \cong A_1 \times ... \times A_n$ is also a principle ideal ring, then each A_i is a principle ideal ring for each $i \in \{1, ..., n\}$. Similarly $\mathbb{Z}_{n_1} \times ... \times \mathbb{Z}_{n_r}$ can be decomposed into the finite product of local rings by the fundamental theorem of finite abelian groups. For both rings of the form $A_1 \times ... \times A_n$ and $\mathbb{Z}_{n_1} \times ... \times \mathbb{Z}_{n_r}$, two ideals are comaximal if their ring contributions are comaximal for each index.

4.4 Graphs of ring products

The previous sections have been concerned with analyzing properties of graphs that represent rings. The purpose of this analysis is to characterize which graphs could be the comaximal ideal graph of a ring. To continue this analysis we consider how the graph of a direct product of finitely many rings relates with the graphs of the individual rings. This relationship is very important, because direct products can be used to build up large complicated rings from simpler and better understood rings. So the following is another approach in understanding the realizability of graphs as comaximal ring graphs.

Recall that the direct product of two graphs $G = (V, E)$ and $G' = (V', E')$ is the graph with vertices $V \times V'$, where $\{(v, v'), (w, w')\}$ is an edge if and only if $\{v, w\} \in E$ and $\{v', w'\} \in E'$ [10]. This graph is denoted $G \times G'$. This section refers to how this graph relates with the graph of a direct product of rings.

Definition 4.4.1. *The ring graph product of two rings graphs $\mathcal{G}(R) \star \mathcal{G}(R')$ is the direct graph product $\widetilde{\mathcal{G}}(R) \times \widetilde{\mathcal{G}}(R')$, plus edges of the form $\{(R, I'), (R, J')\}$ and $\{(I, R'), (J, R')\}$, where I and J are comaximal in R and I' and J' are comaximal in R', and without the vertex $R \times R'$.*

Claim 4.4.2. *The comaximal ring graph $\mathcal{G}(R \times R')$ is equal to the ring graph product $\mathcal{G}(R) \star \mathcal{G}(R')$.*

Proof. Two ideals (I, I') and (J', J') in $R \times R'$ are comaximal if I and J are comaximal in R, and I' and J' are comaximal in R'. This edge condition is equivalent to that of the direct product. But each ring is also comaximal with itself, although this is not seen in the graph $\mathcal{G}(R)$, so we must include edges of the form $\{(R, I'), (R, J')\}$ and $\{(I, R'), (J, R')\}$, where I and J are comaximal in R and I' and J' are comaximal in R'. □

The following is a tool that may be later used in the analysis of how rings can be built up with direct products. Understanding the structure of the graphs equal to the comaximal ideal graph for a ring product may help elucidate what kind of graphs can represent rings.

Proposition 4.4.3. *Let $R = R_1 \times ... \times R_n$. Let e_i denote the number of edges in $\widetilde{\mathcal{G}}(R_i)$. Suppose that e_i is finite for all $i \in \{1, ..., n\}$. Then the number of edges in $\mathcal{G}(R)$ is equal to $\sum_{i=1}^{n} \sum_{j_1<...<j_i \in \{1,...,n\}} e_{j_1}...e_{j_i}$.*

Proof. We know that $\mathcal{G}(R)$ is the ring product graph of the comaximal ring graphs $\mathcal{G}(R_1)$ through $\mathcal{G}(R_n)$. If two ideals $I_1 \times ... \times I_n$ and $J_1 \times ... \times J_n$ are adjacent in this graph then for each $i \in \{1, ..., n\}$ we know $I_r + J_r = R$, which may mean that $I_r = J_r = R$, (although we cannot have $I_r = J_r = R$ for all r). To count the number of edges we will differentiate between I_r, J_r pairs where $I_r = J_r = R$ and where $I_r \neq J_r$. For any pair of comaximal ideals there is some $s \in \{1, ..., n\}$ such that $I_r \neq J_r$ for s indices. We need to count the number of pairs of comaximal ideals, since this is equal to the number of edges. If we restrict ourself to pairs where i many of the pairs have $I_r \neq J_r$, then the number of edges is $\sum_{j_1<...<j_i \in \{1,...,n\}} e_{j_1}...e_{j_i}$. This sum goes over all possible ideal pairs where i ring contribution pairs have $I_r \neq J_r$ pairs. For each set of i ring contributions it finds the number of possible comaximal pairings by taking the product of the number of edges from the graph of each ring contribution where $I_r \neq I_j$. So the number of edges is $\sum_{i=1}^{n} \sum_{j_1<...<j_i \in \{1,...,n\}} e_{j_1}...e_{j_i}$. □

5 Completeness, connectedness, and diameter

The work of Maimani et al., explores restrictions on the completeness, connectedness, and diameter of Γ_2 with a view toward determining which graphs could represent rings. I investigated these graphical properties to see what they can tell us about which graphs can be $\mathcal{G}_2(R)$ for some R.

5.1 Completeness conclusions

Proposition 5.1.1. *The graph $\mathcal{G}_2$ is a complete bipartite graph if and only if $|Max(R)| = 2$.*

Proof. Suppose that $\mathcal{G}_2$ is a complete bipartite graph. By Theorem 4.1.2, it must have precisely two maximal ideals. Now suppose that $|Max(R)| = 2$. Theorem 4.1.2 implies $\mathcal{G}_2$ is bipartite. Let $\mathcal{V}_1, \mathcal{V}_2$ be a partition of the vertices. Then $\mathcal{V}_1$ contains one maximal ideal M_1, and $\mathcal{V}_2$ contains the other maximal ideal M_2. Let I be an arbitrary vertex in $\mathcal{G}_2$. Since I is not contained in the Jacobson radical we know that I is contained in one of the maximal ideals, say M_1, and not contained in the other one. This means that $I \in \mathcal{V}_1$. So every non-maximal ideal is comaximal with one maximal ideal and contained the other maximal ideal (the maximal ideal in its partitioning set). Suppose I_1 and I_2 are contained in partitioning sets $\mathcal{V}_1$ and $\mathcal{V}_2$ respectively. Then $I_1 + I_2 = R$. Indeed, we know $I_1 + I_2$ is not contained in either of the maximal ideals since I_1 and I_2 are contained in distinct maximal ideals. So $I + J = R$. Hence $\mathcal{G}$ is a complete bipartite graph. □

5.2 Connectedness and Diameter Conclusions

Proposition 5.2.1. *Let $R = R_1 \times .... \times R_n$, where each R_i is a local ring and n is a finite positive integer. There exists a vertex v in $\mathcal{G}_2$ which is connected to all other vertices of $\mathcal{G}_2$ if and only if $n = 2$ and R_1 is a field.*

Proof. If we represent the graph as an n-partite graph where n may equal ∞, then we know that one partitioning set must contain only v. So, after possible reordering of the rings in the direct product we know that this vertex is of the form $M_1 \times R_2 \times ... \times R_n$ where M_1 is the maximal ideal of R_1. We will show $n = 2$ by contradiction. Suppose $n > 2$. Then v is not adjacent with $w = M_1 \times 0 \times R_3 \times ... \times R_n$, so w must be contained in the Jacobson radical. But w is not contained in the maximal ideal $R_1 \times R_2 \times M_3 \times R_4 \times ... \times R_n$, where M_3 is the maximal ideal of R_3, which means that w cannot be in the Jacobson radical. Hence we have a contradiction, which means $n = 2$. So we may write $R = R_1 \times R_2$

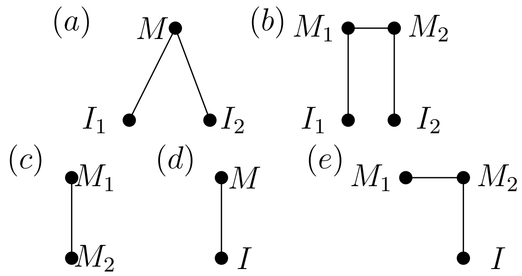
If R_1 is not a field then by Theorem 4.2.2 then v is not comaximal with the ideal $0 \times R_2$ which gives us another contradiction.

It is clear that if $R = R_1 \times R_2$ where R_1 is a field, then $0 \times R_2$ is connected to all other vertices in $\mathcal{G}_2$. □

Note that if R is local then $\mathcal{G}$ contains no edges because all of the ideals are contained in the maximal ideal, which is in this case the Jacobson radical.

Claim 5.2.2. *Suppose R is not local. The graph $\mathcal{G}_2$ is connected, and $diam(\mathcal{G}_2) \leq 3$.*

Proof. In $\mathcal{G}_2$, every ideal is adjacent to a maximal ideal, so because the maximal ideals form a complete subgraph, this graph is connected. We will show that this also means that $diam(\mathcal{G}_2) \leq 3$ with cases. The parenthetical references for each case refer to the diagram below. Suppose non-maximal ideals I_1, I_2 are connected with M_1 and M_2 respectively. If $M_1 = M_2$ then the distance between I_1 and I_2 is 2 (a), otherwise since M_1 and M_2 are connected the distance between I_1 and I_2 is 3 (b). If we have two maximal ideals then they adjacent (c), and if we have a maximal ideal and an ideal not contained in it then the ideals are adjacent (d). Lastly, if we have an ideal I contained in a maximal ideal M, then the distance between I and M is 2. Indeed I is comaximal with a maximal ideal N which is comaximal with M (e).



□

Note that the restrictions in the diameter of the graph are actually a consequence of the more general fact that the ideals form a lattice under containment, and have nothing to do specifically with ring theory. In the following section we will see how lattice structure restricts the diameter of $\mathcal{G}_2$.

6 The application of lattice theory

As shown in section 5.2, many restrictions on the graph $\mathcal{G}_2$ are a consequence of the lattice structure of the ideals. In this section we will consider how we can derive $\mathcal{G}$ from the ideal lattice directly. We will see how we can expand our results by considering graphs derived from lattices which do not represent rings. We begin our discussion of how lattices relate to the ring graph with two equivalent definitions for a lattice.

Definition 6.0.3. *A lattice L is an algebra with two binary operations ($\wedge$ and $\vee$) satisfying for all a, b, c in L the following conditions:*

- *For all a, b, there is a unique $a \wedge b \in L$.*
- *For all a, b, there is a unique $a \vee b \in L$.*
- $a \vee b = b \vee a$
- $a \wedge b = b \wedge a$
- $a \wedge (b \wedge c) = (a \wedge b) \wedge c$
- $a \vee (b \vee c) = (a \vee b) \vee c$
- $a \wedge (b \vee c) = a$
- $a \vee a \wedge b = a.$

Definition 6.0.4. *A lattice is a partially ordered set in which every pair of elements a, b has a greatest lower bound (represented by $a \wedge b$ or glb(a,b)) and a least upper bound (represented by $a \vee b$ or lub(a,b))within the set [6].*

For the ideal lattice $L(R)$ of a ring, the elements of the lattice are ideals, and, for two elements $a, b \in L(R)$ we have $a \vee b = a + b$ and $a \wedge b = a \cap b$. The elements are partially ordered by containment. An apex element is greater than every elements of the lattice, and a nadir element is less than every element of the lattice. In this lattice construction, we will say that for two vertices $I \leq J$, that J is minimal over I if there is no ideal $A \neq I, J$ such that $I \leq A \leq J$. If the apex is minimal over an element M then M is said to be a *maximal element*. We will only consider lattices with at least one maximal element. A *chain* between two vertices $I \leq J$ is a series of elements $v_1, v_2, ..., v_n$ such that $I = v_1 \leq ... \leq v_n = J$. If there are elements in a lattice which is contained in all of the maximal elements, then the largest of these elements, if it exists is called the *Jacobson element* of the lattice.Visually speaking, the least upper bound for two elements a, b is the smallest element c such that $a \leq c$ and $b \leq c$.

6.1 Understanding the ideal structure of a ring in terms of lattices

We can think of the ideal structure of a ring as a lattice based on containment. So if one element I contains another element J, then I is said to be larger and J is said to be smaller. The largest element is always the (1) ideal and the smallest entry is naturally (0). So the lattice always has an apex and a nadir. A lattice is complete if for any arbitrary set of elements there is a meet and a join [7]. So ideal lattices are always complete. Maximal elements in the lattice are maximal ideals in the ring. An element I is contained in the Jacobson radical if, for any maximal ideal M, we have $I \leq M$. Because the intersection of all of the maximal ideals of a ring is an ideal, namely the Jacobson radical, we know that the lattice of a ring has an Jacobson element which is the Jacobson radical $J(R)$.

Definition 6.1.1. *Let L be a lattice with apex R. The comaximal ideal graph $\mathcal{G}(L)$ is constructed as follows. Set a vertex for every element in the lattice excluding R. For elements I and J in the lattice, the two elements are comaximal if the $lub(I, J) = R$. That is, the only element A such that $I, J \leq A$ is R. Note that this construction only requires that the lattice contains an apex element, but to represent a ring it must have a nadir and an Jacobson element.*

Note that this construction only requires that the lattice contains an apex element, but to represent a ring it must have a nadir and an Jacobson element.

Definition 6.1.2. *Let $\mathcal{G}_2(L)$ be the subgraph of $\mathcal{G}(L)$ induced by vertices of degree strictly greater than zero. This subgraph is the comaximal ideal graph of the sublattice of L which excludes elements less than or equal to the Jacobson element.*

Note that if L is actually the ideal lattice of a ring R then $\mathcal{G}(L(R)) = \mathcal{G}(R)$ and $\mathcal{G}_2(L(R)) = \mathcal{G}_2(R)$ The following proposition is useful in understanding how to construct $\mathcal{G}(L)$, when L is an ideal lattice.

Proposition 6.1.3. *Let $R = R_1 \times ... \times R_n$ and $S = S_1 \times ... \times S_n$, be rings which are the direct product of local rings R_i, and S_j. If the ideal lattices for each pair R_i, S_j are isomorphic, then $\mathcal{G}_2(L(R)) \cong \mathcal{G}_2(L(S))$.*

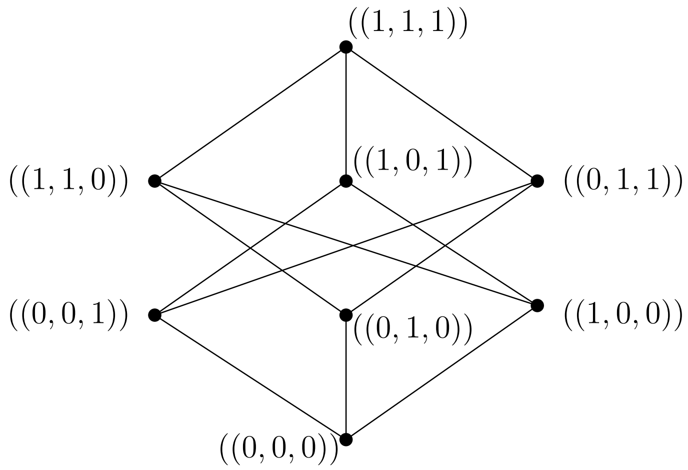
Proof. The proposition is true by construction of $\mathcal{G}(L)$. □

Corollary 6.1.4. *Let $R = R_1 \times ... \times R_m$ and $S = S_1 \times ... \times S_n$, where R_i and S_j are fields for $i \in \{1, ..., m\}$ and $j \in \{1, ..., n\}$. The graphs are isomorphic if and only if $n = m$.*

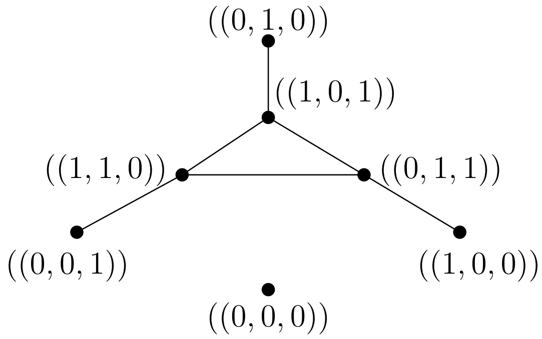
Proof. This corollary is true because fields only have two ideals, (1) and (0). □

Example 6.1.5. *Let $R = F_1 \times F_2 \times F_3$, where F_1, F_2, and F_3 are fields. The following diagram depicts $L(R)$ and $\mathcal{G}(R)$.*

$L(R)$



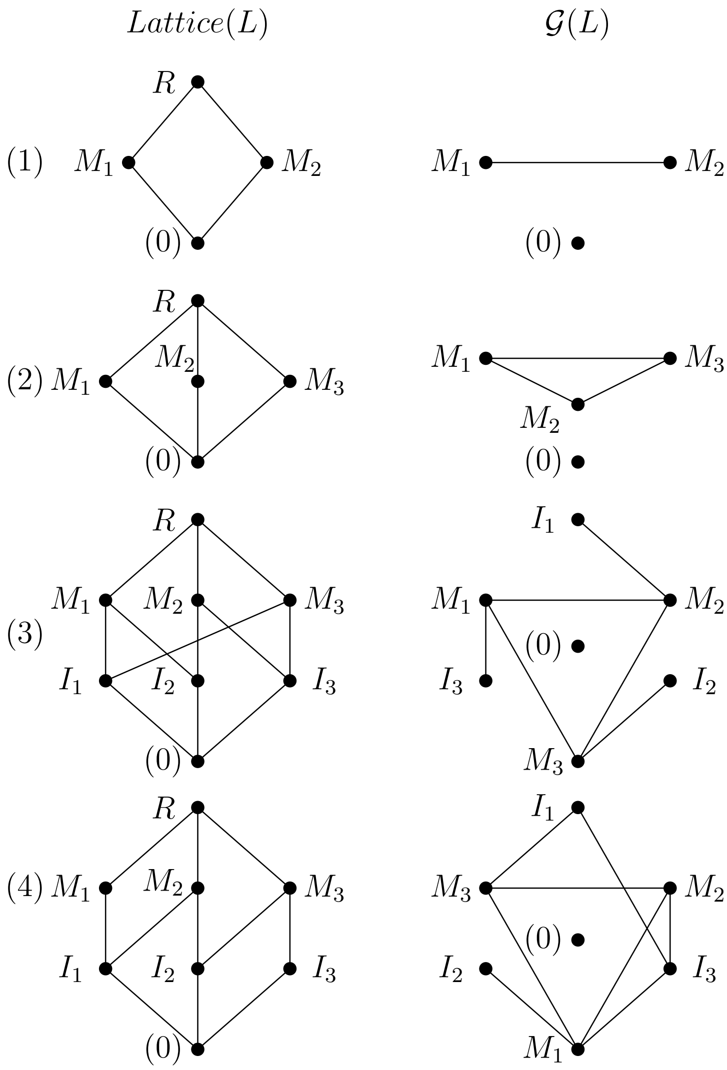
$\mathcal{G}(R)$



Claim 6.1.6. *To show that two ideals are comaximal in a ring R, it suffices to show that their least upper bound in $L(R)$ is (1).*

Proof. Let I_1 and I_2 be two elements in the lattice for a semilocal ring such that $I_1 \vee I_2 = (1)$. So there is no maximal ideal that contains both of the ideals. Because there is no such maximal ideal we know that $I_1 + I_2 \nsubseteq M$ for all maximal ideals M in the ring. So $I_1 + I_2 = R$. $\square$

Example 6.1.7. *These examples show the graphs that result from different lattices.*



6.2 Connectedness and diameter

We proceed by noting several properties of the comaximal ideal graph that are intrinsically a result of lattice theory.

Proposition 6.2.1. *For a lattice L with an apex, the graph $\mathcal{G}_2(L)$ is connected.*

Proof. Let I be a lattice element represented in $\mathcal{G}(L)$. We know I is not contained in an maximal element M since $deg(I) > 0$ in $\mathcal{G}_2(L)$. So every element in the graph is adjacent to a maximal element. Because all of the maximal elements are adjacent by construction, we have that $\mathcal{G}_2(L)$ is connected. $\square$

We will identify certain restrictions on the diameter of $\mathcal{G}_2(L)$ based on the lattice L with apex R. In this section we will deal with both graphs of general lattice which have at the very least, an apex, and graphs of lattices that represent rings. For diameter, we consider only cases where the diameter is at least one. If the diameter is zero, that is, if there is an unconnected vertex, then, the lattice just has one maximal element.

Proposition 6.2.2. *Let L be a lattice with an apex R. The diameter of $\mathcal{G}_2(L)$ is less than or equal to three.*

Proof. Based on the construction of comaximal ideal graphs from a lattice we know that the elements directly less than R are maximal and form a complete subgraph, which has diameter one. Let I an element contained in a maximal element. Then, for a maximal elements M, N, we know $I \leq M$ and $I \not\leq N$, by construction of the lattice. So I and N are adjacent in $\mathcal{G}(L)$. So every non-maximal element is adjacent to a maximal element in $\mathcal{G}(L)$. So for two non-maximal elements I, J we know that I is adjacent with N_1 and J is adjacent with N_2 where N_1 and N_2 are maximal elements. So either I is adjacent with N_1 which is adjacent with N_2 which is adjacent with J. Or if $N_1 = N_2$ then I is adjacent with N_1 which is adjacent with J. Either way the distance between I and J is at most three. □

Claim 6.2.3. *Let L be a lattice with an apex. The graph $\mathcal{G}_2(L)$ has diameter one if and only if, for any element $I \in L$, we have that I is maximal or I is contained in all of the maximal elements.*

Proof. Suppose that $\mathcal{G}_2(L)$ has diameter one. Let I be an element which is not less than a maximal element M. If I is not maximal then I is contained in a maximal element N, so I is not adjacent with N in $\mathcal{G}(R)$. So there can be no non-maximal ideals not contained in all of the maximal ideals. Suppose that for any element I of L we have that I is maximal or $I \subseteq M$ for all maximal elements in L. Because the maximal ideals of a ring form a complete subgraph in $\mathcal{G}_2(R)$ it is clear that the diameter of $\mathcal{G}_2(R)$ is one. □

Note that, by the above claim, we know that any complete graph on n vertices K_n is realizable as the comaximal ideal graph of a lattice. In particular, we know $\mathcal{G}_2(L) = K_n$ if L is the lattice with n maximal elements and an apex and elements less than or equal to the Jacobson element.

Claim 6.2.4. *Let $R = R_1 \times ... \times R_n$ where each R_i is a local ring and $n > 1$. Then the diameter of $\mathcal{G}_2(L(R))$ is one if and only if $n = 2$ and R_1, R_2 are fields.*

Proof. Because $n > 1$ we know that R is not local, so the graph contains at least one edge. If $diam(\mathcal{G}_2) = 1$ then $\mathcal{G}_2$ is a complete graph. In particular it has a vertex connected to all the other vertices, so by Proposition 5.2.1 it is of the form $R_1 \times R_2$ where R_1 is a field and R_2 is a local ring. We know that if I is an ideal strictly contained in M, the maximal ideal of R_2, then $R_1 \times M$ is not adjacent with $R_1 \times I$, so $R_1 \times I$ must be contained in the Jacobson radical. But if $I = (0)$ we know $F \times 0$ is not in the Jacobson radical because it is not contained in the maximal ideal $(0) \times R_2$. So $M = (0)$ and therefore R_2 must be a field. □

Claim 6.2.5. *Let L be a lattice with an apex R. Suppose $\mathcal{G}_2(L)$ has diameter two. If I_1 and I_2 are not maximal elements then there is a maximal element which does not contain either of them or their least upper bound is R.*

Proof. If non maximal elements I_1 and I_2 have least upper bound R the distance between them is one. The distance between two maximal elements is one, and the distance between a maximal element and a non-maximal element is at most 2, as described in Proposition 6.2.2. Let I_1 and I_2 be non-maximal elements in L. Suppose that, for all maximal elements M, we have $I_1 \leq M$ or $I_2 \leq M$. Then there is no path of length two between any I_1 and I_2 in $\mathcal{G}(L)$, so I_1 and I_2 must be adjacent, which means that their least upper bound is R. $\square$

Example 6.2.6. *A graph with diameter two.*

Consider $\mathbb{Z}_2 \times \mathbb{Z}_4$. This graph has diameter two since the distance from (1,0) and (1,2) is two (the maximal ideal ((0,1)) is comaximal with both ideals), but for every pair of non-maximal ideals there is an ideal that contains them.

Many of the above results depended directly on the structure of a lattice than a ring for proof. This dependence suggests that we can extend the results to other mathematical objects which can be represented as a lattice by containment, such as subgroups of a group. In the future it would be interesting to see how these results extend.

6.3 Lattices and graph products

We next consider how $L(R \times R')$ looks in comparison with $L(R)$ and $L(R')$ for two rings R, R'. Understanding this comparison is analogous to understanding how $\mathcal{G}(R \times R')$ relates with $\mathcal{G}(R)$ and $\mathcal{G}(R')$. It helps us consider the structure of rings which are the finite direct product of simple rings.

Definition 6.3.1. *The cartesian product of two graphs $G = (V, E)$ and $G' = (V', E')$ is the graph $G \Box G'$ where the vertex set is $V \times V'$ and $(v, v'), (w, w')$ is an edge if and only if $v = w$ and $v', w' \in E'$, or $v' = w'$ and $v, w \in E'$ [10].*

Proposition 6.3.2. *The lattice for the direct product of two rings R and R' is equal to the cartesian product of the two lattices.*

Proof. It suffices to find out when two elements of the lattice $I \times I'$ and $J \times J'$ are adjacent. Based on the construction of a lattice we know either $I \times I' \leq J \times J'$ or $J \times J' \leq I \times I'$. Without loss of generality suppose that $I \times I' \leq J \times J'$. Then $I \leq J$ and $I' = J'$, or $I = J$ and $I' \leq J'$. These conditions correspond with the definition of the cartesian product. $\square$

6.4 Diameter of graph products

Now that we have considered how the structure of a lattice affects its diameter, we consider the diameter of the graph of the product of two lattices, as a result of the original lattice graph diameters.

Proposition 6.4.1. *Let L, L' be lattices with apices R, R' and nadirs Z, Z' such that $\mathcal{G}_2(L)$ and $\mathcal{G}_2(L')$ have diameter zero. Then the diameter of $\mathcal{G}_2(L \Box L')$ is one if both lattices have only maximal elements and an apex. The diameter of $\mathcal{G}_2 \Box \mathcal{G}_2'$ is two if at least one of L and L' has a non-maximal element.*

Proof. Clearly each lattice has just one maximal element. If there are no non-maximal elements other than the apex in each lattice, then the diameter is one. Otherwise one lattice, without loss of generality say L', has a non maximal element $I \neq Z'$. So the distance between, $R \times I$ and $R \times Z'$ is at least two. However the diameter of this graph is at most two. Let $I_1 \times J_1$ and $I_2 \times J_2$ be two non-comaximal elements which are not adjacent in $\mathcal{G}_2(L \Box L)$. The maximal elements of $L \Box L'$ are (R, M') and (M, R'). Either $I_1 = R$ or $J_1 = R'$, or $I_1 \times J_1$ would be less than both maximal elements. Without loss of generality suppose $I_1 = R$. Similarly one of $I_2 = R$ or $J_2 = R'$ is equal to (1). but since $I_1 \times J_1$ and $I_2 \times J_2$ are not adjacent this must be I_1. Then $(0) \times R'$ is not larger than either of these ideals so the distance between them is two. □

Proposition 6.4.2. *Let L, L' be lattices with apices R, R' such that $\mathcal{G}_2(L)$ has diameter one and $\mathcal{G}_2(L')$ has any diameter and a nadir Z. Suppose that L has at most two maximal elements. Then the diameter of $\mathcal{G}_2(L \Box L')$ is three.*

Proof. Because $\mathcal{G}_2(L)$ has diameter one we know that it contains at least two maximal elements, M_1 and M_2. For L' we just know that R' has R' and Z as elements. So $\mathcal{G}_2(L \Box L')$ has the elements (M_1, Z), (M_2, R'), (M_1, R'), and (M_2, Z). Clearly $lub((M_1, Z), (M_1, R')) = lub(M_2, R'), (M_1, R')) = lub((M_2, R'), (M_2, Z)) = (R, R')$. However $lub((M_1, Z), (M_1, R')) = (M_2, R') \neq (R, R')$, $lub((M_2, R'), (M_2, Z)) = (M_2, R') \neq (R, R')$, $(M_1, Z), (M_2, Z)) = (R, Z) \neq (R, R')$. So unless there is another maximal element not containing either (M_1, Z) or (M_2, Z), the diameter of $\mathcal{G}_2(L \Box L)$ is three. Such an element would have to be of the form (M, R') where $M \neq M_1, M_2$ is maximal in L. There is no such maximal element, so the diameter is three. □

Note that if the lattice in the above proof represented a ring, then L has at most two maximal elements. This will be proven in section 6.5.

Proposition 6.4.3. *Let L, L' be lattices with apices R and R' such that $\mathcal{G}_2(L)$ has diameter three and $\mathcal{G}_2(L')$ has any diameter and a nadir Z. Then the diameter of $\mathcal{G}_2(L \Box L')$ is three.*

Proof. Because $\mathcal{G}_2(L)$ has diameter three we know that it contains at least two elements I_1, I_2 which are three edges apart. So every maximal element is larger than at least one of them, and each of I_1, I_2 is not less than in maximal elements M_1 and M_2 respectively. So we know I_1 is adjacent with M_1 and I_2 is adjacent with M_2 in $\mathcal{G}_2(L)$. We also know that $R' \geq R', Z$.

We know $\mathcal{G}_2(L \Box L')$ includes the elements (I_1, Z), (M_1, R'), (M_2, R'), and (I_2, Z). Clearly $lub((I_1, (0)), (M_1, R')) = lub((M_2, R'), (M_1, R')) = lub((M_2, R'), (I_2, Z)) = (R, R')$. However $lub((I_1, Z), (M_1, R')) = (M_1, R') \neq (R, R')$, $lub((M_2, R'), (I_2, Z)) = (M_2, R') \neq (R, R')$, and $(I_1, Z), (I_2, Z) \neq (R, R')$. So unless there is a maximal element which is not larger than either (I_1, Z) or (I_2, Z), the diameter of this graph is three. Such an element would have to be of the form (M, R'), with M not larger than either I_1 or I_2. Because there is no such element in $\mathcal{G}_2(L \times L)$ the diameter is three. □

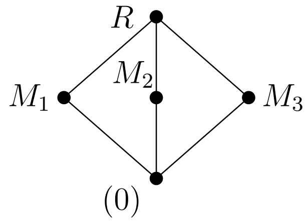
Proposition 6.4.4. *Let L, L' be lattices with apices R, R' such that $\mathcal{G}_2(L)$ has diameter two and $\mathcal{G}_2(L')$ has diameter zero or two. Then the diameter of $\mathcal{G}_2(L \Box L')$ is two.*

Proof. Because $\mathcal{G}_2(L)$ has diameter two we know that it has a non-maximal ideal I which is not contained in a maximal element $M_1 \in L$, but for which there is $J \in L$ such that $lub(I, J) \neq R$ and J is not contained in M_1. Then $lub((I, R'), (M_1, R')) = (R, R') = lub((M_1, R'), (J, R'))$, but $lub((I, R'), (J, R')) = (J, R') \neq (R, R')$, so the diameter is at least two. Suppose we have two non-maximal elements (I_1, J_1) and (I_2, J_2) which are not adjacent in $\mathcal{G}_2(L\Box L')$. If $lub(I_1, I_2) \neq R$ then there is $(M \in R)$ which does not contain I_1 or I_2. So (M, R') is comaximal with both (I_1, J_1) and (I_2, J_2). If $lub(J_1, J_2) \neq R'$ then there is $(N \in R')$ which does not contain J_1 or J_2. So (R, N) is comaximal with both (I_1, J_1) and (I_2, J_2). So the diameter is at most two. $\square$

Based on the above results, it seems that most lattice product graphs end up having diameter three, so diameter three is probably the most common diameter. It would be interesting to investigate this in the future.

6.5 An impossible lattice

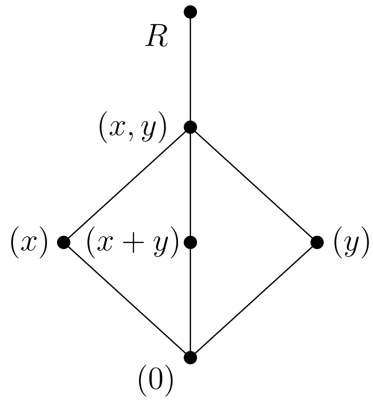
Considering various constructions of lattices whose graphs have diameter one, two, or three suggests that it is relevant to consider which of these lattices could represent actual rings. To that end we consider the following impossible lattice which cannot represent a ring. This lattice was specifically referenced in relation with Proposition 6.4.2.



Claim 6.5.1. *There is no ring R that only contains three maximal ideals, the trivial ideal, and the ring itself as ideals.*

Proof. Suppose that such a ring exists. Then the three maximal ideals, which we may denote M_1, M_2, and M_3, must be principal ideals. If not, then one of the maximal ideals would have at least two generators g_1, g_2, and both (g_1) and (g_2) would be non-maximal and nontrivial ideals. So $M_1 = (a)$, $M_2 = (b)$, and $M_3 = (c)$ for some elements a, b, c. We know that $ab = ac = bc = 0$, because all of the maximal ideals are comaximal to each other, and if two ideals are comaximal then their product equals their intersection. Because (b) and (c) are comaximal we know that we can write $a = rb + sc$, for $r, s \in R$. So $a^2 = a(rb + sc) = 0$. Moreover we know we have $1 = av + bw$, for $v, w \in R$, so $a = a^2v + abw = 0v + 0w = 0$, which is a contradiction. $\square$

This claim shows that the above lattice cannot depict an actual ring. Note that the above lattice, which is modular but not distributive, can actually be contained as a sublattice in the lattice of a ring. Since such an embedding means that the lattice is modular but not distributive [6], comaximal ideal graphs may be derived from rings with lattices that are modular. For example we have the ring $R = \mathbb{F}_2[x, y]/(x, y)^2$ with the following lattice:



6.6 Graphs which can be derived from a lattice

Because certain lattices cannot represent rings, it is natural to question what graphs cannot be the comaximal ring graph for a lattice. To approach this question I considered variations on connectivity of possible $\mathcal{G}_2$ graphs and found examples for which it is impossible to depict a lattice.

Proposition 6.6.1. *Suppose we can know which vertices in $\mathcal{G}_2$ are maximal. Then for every non-maximal ideal I there is a maximal ideal vertex M which I is not adjacent to and a different maximal ideal N which I is adjacent to. Furthermore the diameter of the graph should be less than or equal to three*

Proof. This comes from the definition of $\mathcal{G}_2$ and Proposition 6.2.2. Note that if R is local then the only vertex in $\mathcal{G}_2(R)$, the maximal ideal. $\square$

For the above claim, we can locate all of the maximal ideals if $\mathcal{G}_2$ is n-partite and for there is a unique vertex of highest degree in each partitioning set.

Proposition 6.6.2. *Consider $\mathcal{G}_2(R)$ for a semilocal ring R. We know the graph is n-partite. Let J_1, J_2 be two non-maximal ideals, and let $\mathcal{M}_1$ and $\mathcal{M}_2$ be the respective sets of maximal ideals that J_1 and J_2 are comaximal with. If $\mathcal{M}_1$ and $\mathcal{M}_2$ are disjoint then J_1 and J_2 are comaximal and should therefore be adjacent.*

Proof. This comes from the construction of $\mathcal{G}_2$ from a lattice. $\square$

Proposition 6.6.3. *Let R be a ring with a finite number of ideals. Let J_1, J_2 be two non-maximal ideals. Let $\mathcal{N}_1$ and $\mathcal{N}_2$ be the respective sets of maximal ideals which they are not comaximal with. If J_1 and J_2 are comaximal, then $\mathcal{N}_1$ and $\mathcal{N}_2$ must be disjoint.*

Proof. Suppose there are non-maximal ideals J_1, J_2 in R such that $J_1 + J_2 = R$. Suppose there is a maximal ideal M such that $J_1 + M \subset R$ and $J_2 + M \subset R$. Then $J_1, J_2 \subseteq M$. Hence $J_1 + J_2 \subseteq M \subset R$, and we have a contradiction. $\square$

Note that in a ring with finitely many ideals, if J_1 and J_2 are comaximal non-maximal ideals, then there may be a maximal ideal M such that $J_1+M = J_2+M = R$. For instance in $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ the ideals $((1,0,1,0,1))$ and $((0,1,0,1,1))$ are comaximal with each other and are both comaximal with the maximal ideal $((1,1,1,1,0))$.

In fact we can find many such pairs of non-maximal ideals who are comaximal to common maximal ideals.

Theorem 6.6.4. *If a ring R is equal to a finite product of n fields, for any maximal ideal M there are $(\binom{n-1}{2}\binom{n-3}{2}\sum_{k=0}^{n-5}\binom{n-5}{k}\sum_{l=0}^{n-k-5}\binom{n-k-5}{l})/2$ non-maximal ideal pairs which are both comaximal to M.*

Proof. Recall that any ideal I in $R = F_1 \times ... \times F_n$ can be denoted $I_1 \times ... \times I_n$ where I_i is F_i or (0). To find pairs of non-maximal ideals that are comaximal to the same ideal we begin by choosing an arbitrary maximal ideal M. We know M has only one of its ring contributions to be (0), without loss of generality let this be the ring contribution from F_n. So both of the ideals have F_n as the ring contribution from F_n. If two non-maximal ideals I, I' are comaximal then if one has a ring contribution of (0) for the ith field then the other has F_i as a ring contribution. So the choice of one ideal limits the possibilities for the second. Moreover each ideal needs at least two ring contributions to be zero, so that they are not maximal. Thus we begin by choosing two rings to contribute (0) from the remaining $n-1$ options for I, and then two more to contribute (0) from the remaining $n-3$ options. So there are $n-5$ remaining ring contributions to assign for I. We can choose k many of them to be 0, for k between $n-5$ and 0, and then set the rest to be (1). Then for I', those k positions are given the full field as a ring contribution, and the remaining $n-k-5$ slots can be zeros or ones. So for l between $n-k-5$ and 0, have l of the remaining ring contributions in I' be zero. We divide by two because this counting technique double counts the number of ideals.

□

7 Relating isomorphisms between graph types

It is clear that if two rings are isomorphic then they will have isomorphic graphs (because the rings will have the same ideal structure). Our question is: what do isomorphisms between graphs tell us about the rings they represent? We approach this question by analyzing variations on graphical representations of R, more specifically by comparing $\mathcal{G}$ with $\mathcal{G}_2$ and $\mathcal{G}(R/J(R))$.

7.1 Isomorphisms between $\mathcal{G}_2$ graphs

Theorem 7.1.1. *Let R and S be two semilocal rings with m and n many maximal ideals, respectively. Suppose that $\mathcal{G}(R) \cong \mathcal{G}(S)$. Then $m = n$ and there is an isomorphism f between $\mathcal{G}(R)$ and $\mathcal{G}(S)$ with the following property: if M is maximal in R, then $N = f(M)$ is maximal in S.*

Proof. By Theorem 4.1.2 we know that $\mathcal{G}(R)$ and $\mathcal{G}(S)$ are m-partite and n-partite respectively, so we must have $m = n$. Let $f : \mathcal{G}(R) \to \mathcal{G}(S)$ be an isomorphism between the two graphs. Since $\mathcal{G}(R)$ and $\mathcal{G}(S)$ are both n-partite we can take an n-partition of $\mathcal{G}(R)$, denoted $\{\mathcal{V}_1, ..., \mathcal{V}_n\}$ and we know that $\{f(\mathcal{V}_1), ..., f(\mathcal{V}_n)\}$ is a proper n-partition of $\mathcal{G}(S)$.

Recall that each partitioning set contains precisely one maximal ideal, and the maximal ideal is one of the ideals with highest degree for that partitioning set by Lemma 4.1.6. By Proposition 4.1.8 we can identify the maximal ideal in each partitioning set up to equivalence. Let $v_1 \in \mathcal{V}_i$ be a vertex of highest degree. Then $f(v_1)$ is a vertex of highest degree in $f(\mathcal{V}_i)$. Suppose there is another vertex $v_2 \in \mathcal{V}_i$ with the same degree as v_1. Then the map $f' : \mathcal{G}(R) \to \mathcal{G}(S)$ such that $f'(v_1) = f(v_2)$, $f'(v_2) = f(v_1)$, and for $w \neq v_1, v_2 \in \mathcal{G}(R)$ $f'(w) = f(w)$, is also an isomorphism. So beginning with an arbitrary isomorphism between the graphs $\mathcal{G}(R)$ and $\mathcal{G}(S)$, the isomorphism can be modified so that, if M is maximal in R, then $N = f(R)$ is a maximal ideal in S.

In particular, if we know which vertices in $\mathcal{G}(R)$ correspond to maximal ideals in each graph we can construct an isomorphism f' between the graphs such that the maximal ideals in R are sent by f' to maximal ideals of S. For each partitioning set $\mathcal{V}_i$ begin by locating the maximal ideal x. If $f(x)$ is the maximal ideal in $f(\mathcal{V}_i)$, then for each $v \in \mathcal{V}_i$ set $f'(v) = f(v)$. Otherwise set $f(x)$ to be the maximal ideal y of S, set $(f')^{-1}(f(y)) = f(x)$, and for $w \neq x, f^{-1}(y) \in \mathcal{V}_i$ set $f'(w) = f(w)$. $\square$

Proposition 7.1.2. *Let $R = R_1 \times ... \times R_m$, where R_i are local rings with maximal ideals M_i, for $i \in \{1, ..., m\}$. Define c_I to be the number of ideals that are not contained in the Jacobson radical and which are contained in I. Then for $M = R_1 \times ... \times R_{i-1} \times M_i \times R_{i+1} \times .... \times R_m$, we have $c_M = (a_1 \cdot ... \cdot a_{i-1} \cdot (a_i - 1) \cdot a_{i+1} \cdot ... \cdot a_m - 1) - ((a_1 - 1) \cdot ... \cdot (a_{i-1} - 1) \cdot (a_i - 1) \cdot (a_{i+1} - 1) \cdot ... \cdot (a_m - 1))$, where a_k is the number of ideals in R_k.*

Proof. For a maximal ideal, the number of ideals it contains which are not contained in the Jacobson radical equals the number of vertices it is not adjacent with in $\mathcal{G}_2$. We know that the maximal ideal in each partitioning set $\mathcal{V}_i$ is one of the vertices of maximal degree in $\mathcal{V}_i$. Without loss of generality say it is the vertex M of highest degree. The Jacobson radical of R is $M_1 \times ... \times M_m$. For the maximal ideal M we know c_M counts all ideals of the form $(I_1 \times ... \times I_{i-1} \times I_i \times I_{i+1} \times ... \times I_m)$ where, for $j \neq i$, each I_j is an ideal of R_j and at least one of the I_j is actually R_j. The ideal I_i can be any ideal contained in M_i. If all of the $I_j = R_j$ for $j \neq i$, then the ideal is contained in the Jacobson radical, so it is not counted by c_M. If a_k is the number of ideals in R_k for any $k \in \{1, ..., m\}$, the number of ideals contained in $M = R_1 \times ... \times R_{i-1} \times M_i \times R_{i+1} \times .... \times R_m$ equals $a_1 \cdot ... \cdot a_{i-1} \cdot (a_i - 1) \cdot a_{i+1} \cdot ... \cdot a_m$. The number of ideals contained in M and the Jacobson radical is $(a_1 - 1) \cdot ... \cdot (a_m - 1)$. So $c_M = (a_1 \cdot ... \cdot a_{i-1} \cdot (a_i - 1) \cdot a_{i+1} \cdot ... \cdot a_m - 1) - ((a_1 - 1) \cdot ... \cdot (a_{i-1} - 1) \cdot (a_i - 1) \cdot (a_{i+1} - 1) \cdot ... \cdot (a_m - 1))$. This number equals the number of ideals contained in M minus the number of ideals contained in the Jacobson radical. $\square$

Theorem 7.1.3. *Suppose that $R = R_1 \times ... \times R_m$ and $S = S_1 \times ... \times S_n$, where R_i and S_j are local rings for $i \in \{1, ..., m\}$ and $j \in \{1, ..., n\}$ with maximal ideals M_i and N_j respectively, for $m, n > 1$. Then $\mathcal{G}(R) \cong \mathcal{G}(S)$ if and only if $\mathcal{G}_2(R) \cong \mathcal{G}_2(S)$.*

Proof. Suppose that $\mathcal{G}(R) \cong \mathcal{G}(S)$. By Claim 7.1.1 we have $m = n$. By Lemma 3.0.2 we know that for $\mathcal{G}$, the set of vertices in $\mathcal{J}$ is precisely the set of vertices of degree zero. So we must have $|\mathcal{J}(R)| = |\mathcal{J}(S)|$ and $\mathcal{G}_2(R) = \mathcal{G}(R) \setminus \mathcal{J}(R) \cong \mathcal{G}(S) \setminus \mathcal{J}(S) = \mathcal{G}_2(S)$.

Suppose that $\mathcal{G}_2(R) \cong \mathcal{G}_2(S)$. I claim that with $\mathcal{G}_2$ we can determine the vertices required to depict $\mathcal{G}$. In particular, the subgraph $\mathcal{G}'$ from definition 4.2.4 will help us determine which vertices are in $\mathcal{J}$. Without loss of generality we will deal with the graph and corresponding subgraphs for R. Suppose that $I_1 \times I_2 \times ... \times I_n \in \mathcal{J}$. Then $I_i \subseteq M_i$ for all $i \in \{1, ..., n\}$. Recall that the vertices of $\mathcal{G}'$ are of the form $I_1 \times R_2 \times ... \times R_n, ..., R_1 \times R_2 \times ... \times R_{n-1} \times I_n$ for ideals $I_i \neq R_i$ in R_i, as described in section 4.2. For each ideal $I \in J(R)$ we can find a corresponding set of vertices in $\mathcal{G}'$ as follows. Given $I = I_1 \times ... \times I_n$, define $S_I = \{I_1 \times R_2 \times ... \times R_n, ..., R_1 \times R_2 \times ... \times R_{n-1} \times I_n\}$. An element of $\mathcal{S}$ is a clique of size n in $\mathcal{G}'$, which is any set of the form $I_1 \times R_2 \times ... \times R_n, ..., R_1 \times R_2 \times ... \times R_{n-1} \times I_n$ for $I_i \neq R_i$. For a such a set S_I in $\mathcal{S}$, the corresponding ideal is $I_1 \times ... \times I_n$. If $I = I_1 \times ... \times I_n \neq J = J_1 \times ... \times J_n$, then $I_k \neq J_k$ for some $k \in \{1, ..., n\}$, so $S_I \neq S_k$. Hence $S_{I I \in J(R)} = \mathcal{S}$. So there is a one to one correspondence between $\mathcal{J}(\mathcal{R})$ and $\{S_I\}_{I \in \mathcal{J}(\mathcal{R})}$. Note that $\mathcal{G}'$ is a subgraph of $\mathcal{G}_2(R)$ which is easily isolated from the whole graph because it is precisely union of all complete n-partite subgraphs in $\mathcal{G}_2(R)$. Because there is an element in $\{S_I\}$, there is an element in the Jacobson radical for every clique of size n in $\mathcal{G}_2$.

Recall that $K_{r_1,...,r_n}$ is the complete n-partite graph with r_i vertices in the ith partition for each i. The subgraph $\mathcal{G}'$ is a $K_{r_1,...,r_n}$ graph for some $r_1, ..., r_n \in \mathbb{Z}$. So there are exactly $r_1 \cdot ... \cdot r_n$ singletons in $\mathcal{G}$, because n-sets are formed by taking representatives from each partitioning set. So we know how many vertices are in $\mathcal{J}(R)$ and $\mathcal{J}(S)$ because the number of ideals in each is determined by $\mathcal{G}_2(R)$ and $\mathcal{G}_2(S)$ respectively. Because the two are isomorphic, we know $|\mathcal{J}(R)| = |\mathcal{J}(S)|$. Since $\mathcal{G} = \mathcal{G}_2 \cup \mathcal{J}$ we know that $\mathcal{G}(R) \cong \mathcal{G}(S)$ □

7.2 Isomorphisms between $\mathcal{G}(R/J(R))$ graphs

Because $\mathcal{G}$ and $\mathcal{G}_2$ are related, at least for semilocal rings which are the product of finitely many local rings, it is natural to question how $\mathcal{G}(R)$ and $\mathcal{G}(R/J(R))$ relate.

Proposition 7.2.1. *Let R be a ring. Then there is a subgraph of $\mathcal{G}(R)$ isomorphic to $\mathcal{G}(R/J(R))$.*

Proof. There is a one to one correspondence that preserves comaximality between the ideals containing J in R and the ideals of $R/J(R)$ [3]. Consider the induced subgraph $\widehat{\mathcal{G}}$ of $\mathcal{G}$ composed of ideals which contain the Jacobson radical. This subgraph contains precisely the vertices which correspond to the ideals in $R/J(R)$. So $\widehat{\mathcal{G}}$ is isomorphic to $\mathcal{G}(R/J(R))$. □

Example 7.2.2. *If $R = R_1 \times ... \times R_m$ where R_i is local with maximal ideal M_i for $i \in \{1, ..., m\}$, then the set of ideals corresponding to $\mathcal{G}(R/J(R))$ is the set of ideals of the form $I = I_1 \times ... \times I_m$ where $I_i = R_i$ or M_i.*

Proposition 7.2.3. *For Artin rings R and S, if $\mathcal{G}(R) \cong \mathcal{G}(S)$ then $\mathcal{G}(R/J(R)) \cong \mathcal{G}(S/J(S))$.*

Proof. Because R and S are Artin we know $R \cong R_1 \times ... \times R_m$ and $S \cong S_1 \times ... \times S_n$, where R_i, S_j are local Artin rings, for $i \in \{1, ..., m\}$ and $j \in \{1, ..., n\}$. We know by claim 7.1.1 that $m = n$. Every commutative Artin ring $A = A_1 \times ... \times A_r$ modulo its Jacobson radical is isomorphic to a finite direct product of r many local Artin rings. So $R/J(R)$ and $S/J(S)$ are both the direct product of n fields. By Corollary 6.1.4 this means that $\mathcal{G}(R/J(R)) \cong \mathcal{G}(S/J(S))$. □

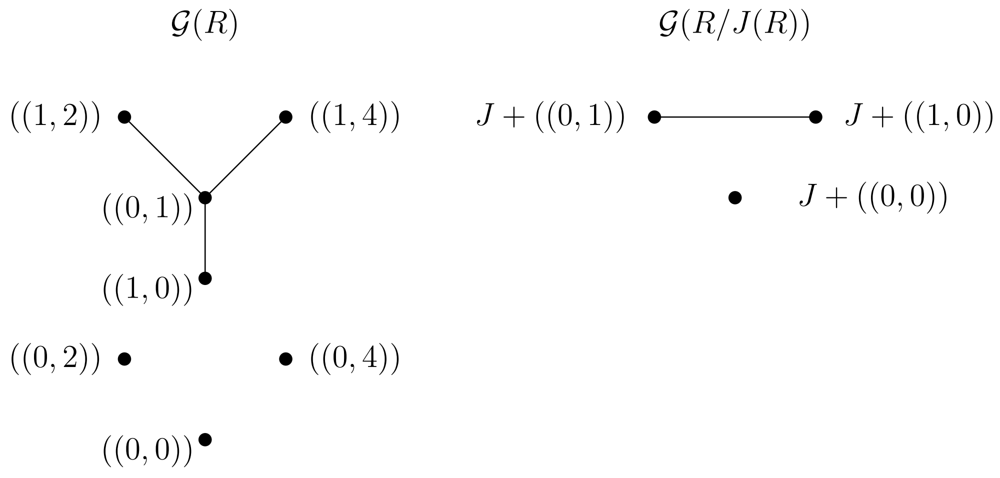
It is natural to ask how $\mathcal{G}(R/J(R))$ and $\mathcal{G}_2$ relate. In the following example we show that the two may not be isomorphic.

Example 7.2.4. *A ring such that* $\mathcal{G}(R/J(R)) \not\cong \mathcal{G}_2(R)$

Take $R = \mathbb{Z}_2 \times \mathbb{Z}_8$. The graph $\mathcal{G}_2(R)$ has edges between ((1,2)) and ((0,1)), ((1,4)) and ((0,1)), and ((1,0)) and ((0,1)). Because $J = ((0,2))$ we know $R/J(R)$ has four cosets:

1. {(0,0), (0,2), (0,4), (0,6)}
2. {(0,1), (0,3), (0,5), (0,7)}
3. {(1,0), (1,2), (1,4), (1,6)}
4. {(1,1), (1,3), (1,5), (1,7)}.

These cosets form the ideals: {(0,0)+J}, $I_1 = \{(0,0)+J, (0,1)+J\}$, $I_2 = \{(0,0)+J, (1,0)+J\}$, and $\{(0,0)+J, (1,0)+J, (0,1)+J, (1,1)+J\} = R/J(R)$. We know that I_1 and I_2 are the only comaximal pair in the graph. So because $\mathcal{G}(R/J(R))$ has one edge while $\mathcal{G}(R)$ has three it is clear that the two graphs are not isomorphic. This can be seen in the following diagram.



8 Connections with Clean Rings

Definition 8.0.5. *A ring is clean if each of its elements can be written as the sum of a unit and an idempotent.*

Note that any local ring is clean, and by Anderson and Camillo the direct product of clean rings is clean [1]. The following results are analogous to Lemma 2.1, in Sharma and Bhatwadekar's research and Theorem 2.5 in Maimani et al.'s research [11, Lemma 2.1],[9, Theorem 2.5].

Lemma 8.0.6. *If R is a ring with an infinite number of idempotents then $\mathcal{G}_2$ contains an infinite clique formed with ideals generated by idempotents.*

Proof. Suppose R has an infinite number of idempotents. Take $e \neq 0, 1$ be a non-trivial idempotent. Then $R = (e) \times (1 - e)$. Because R has infinitely many idempotents this means that at least one of (e) and $(1 - e)$ has infinitely many idempotents. Let (f_1) be the ideal with infinitely many idempotents, or an arbitrary one of the ideals if they both have infinitely many idempotents. By the same process we can find (f_2), an ideal in (f_1) with infinitely many idempotents. Continuing this process we have a chain $(f_1) \subset (f_2) \subset (f_3) \subset ...$, where each (f_n) has infinitely many idempotents and is strictly contained in (f_{n-1}). According to Sharma and Bhatwadekar, if we set $f_0 = 1$ and $e_i{=}f_{i+1} + (1 - f_1)$ for $i \geq 0$, these $e_i's$ are non-trivial idempotents such that $f_n = \prod_{j=1}^{n} e_j$ and (f_n) and (e_{n+1}) are comaximal and they form an infinite clique [11]. □

Theorem 8.0.7. *The ring R is a finite product of local rings if and only if R is clean and clique$(\mathcal{G}_2)$ is finite.*

Proof. Suppose that R is a finite product of local rings: $R_1 \times ... \times R_n$. By Corollary 4.1.3 we know that since R is semilocal it has a finite clique. This ring is also clean since it is the direct product of three rings.

Suppose that the clique size is finite and equal to n. By the Lemma 8.0.6, the finite clique size implies it must have finitely many idempotents. But if R is not the finite product of local rings then R has infinitely many idempotents. Thus we have a contradiction. □

9 Questions for the future

This section covers questions which I found intriguing throughout the semester, but did not have the times to thoroughly investigate. These are questions which I was interested in and which could be avenues for future research.

1. When considering the properties of graphs with diameter one, two, or three, it becomes natural to question the prevalence for certain graph types. Restricting to the case where a ring is isomorphic to the product of n local rings, there is only one graph with diameter one, and graphs with diameter two are rare if n is at least 3.

For instance if $R = R_1 \times R_2 \times R_3$, and R_3 contains a nontrivial, proper ideal I then $(0) \times R_2 \times I$ and $(0) \times (0) \times R_3$ are three apart. So an important question is: what graphical diameters are more common? We can also question the edge density of graphs with diameter two, and the proportion of sparse graphs versus dense graphs.

2. Based on the importance of the lattice structure, it could be interesting to compare what properties distributive verses non-distributive lattices have.

3. How do $\mathcal{G}(R/J(R))$ and $\mathcal{G}_2(R)$ relate, as in example 7.2.4?

4. Another point of consideration for a lattice L with an apex is the structure of the subgraph made up of non-maximal ideals within $\mathcal{G}(L)$. For instance, in a lattice where no ideal is contained in more than one maximal idea,l the subgraph is complete. This graph is also probably not connected.

5. In the section about lattices, we limited our considerations to lattices which have at least one maximal element and an apex. However the definition of $\mathcal{G}(L)$ does not rely on the existence of maximal elements, so it could be interesting to see what kinds of graphs can be formed with lattices without maximal elements.

10 Appendix

We will go over basic definitions and results from graph theory, ring theory, and lattice theory.

10.1 Graph Theory

Definitions 1. *A graph $G = (V, E)$ consists of a set $\mathcal{V}$ of objects called vertices and a set $\mathcal{E}$ of sets of two vertices called edges. If two vertices v_1 and v_2 are connected by an edge then they are called adjacent and the edge between them is denoted v_1, v_2 [5].*

A subgraph for a graph $G = (V, E)$ is a graph $H = (V', E')$ such that $V' \subseteq V, E' \subseteq E$, and all edges in E' are between vertices in V'. The subgraph H is said to be induced by V' if every edge between vertices in V' in E is contained in E'.

A path is a finite ordered set of vertices $\{v_1, v_2, ..., v_n\}$ such that $\{v_i, v_{i+1}\}$ is an edge for $i \in \{1, 2, ..., n-1\}$. A graph is connected if between every pair of vertices $v, w \in \mathcal{V}$ there is a path $\{v_1, v_2, ..., v_n\}$ such that $v = v_1$ and $w = v_n$. The number of edges in the shortest such path is the the distance between the two vertices. The diameter of a connected graph is the supremum of the distances between the vertices (the largest distance between pairs of vertices).

A graph is complete if $\{v, w\}$ is an edge for every pair of vertices $v, w \in \mathcal{V}$. Such a graph with n vertices is denoted K_n.

A graph is n-partite if we can partition $\mathcal{V}$ into n subsets, $\mathcal{V}_1, ..., \mathcal{V}_n$ such that for all $i \in \{1, ..., n\}$, and for any pair $v, w \in \mathcal{V}_i$ we have that $\{v, w\}$ is not an edge. Such a graph *is a complete n-partite graph if we also have that $\{v, w\}$ is an edge if $v \in \mathcal{V}_i$ and $w \in \mathcal{V}_j$ with $i \neq j \in \{1, ..., n\}$. A complete bipartite, or 2-partite, graph is denoted a $K_{m,n}$ graph, where $m = \mathcal{V}_1$ and $n = \mathcal{V}_2$. A complete n-partite graph is denoted $K_{r_1,...,r_n}$, where r_i is the number of vertices in $\mathcal{V}_i$.*

A proper coloring of a graph is an assignment of colors to the vertices such that no two adjacent vertices have the same color. The chromatic number $\chi\{G\}$ of a graph G is the smallest number of colors needed for a proper coloring.

A clique of a graph is a maximal complete subgraph. That is, a complete subgraph induced by a set of vertices to which no vertex may be added such that the subgraph retains its completeness. The number of vertices in the largest clique is the clique number, denoted clique(G).

An isomorphism between two graphs is a map between the vertex sets which is edge preserving. Two graphs are isomorphic if there is an isomorphism between them.

Theorem 10.1.1. *A graph is n-partite if and only if its chromatic number is n. [5]*

There are four kinds of graphical products for two graphs $G = (V, E)$ and $G' = (V', E')$ [10]. For each of these products the vertex set is $V \times V'$, but the edge set differs.

- **Direct Product:** In the direct product $G \times G'$ we have that $[(v, v'), (w, w')]$ is an edge if and only if $[v, w] \in E$ and $[v', w'] \in E'$.
- **Cartesian Product:** In the cartesian product $G \square G'$ we have that $[(v, v'), (w, w')]$ is an edge if and only if $v = w$ and $[v', w'] \in E'$, or $v' = w'$ and $[v, w] \in E'$.
- **Strong Product:** In the strong product $G \boxtimes G'$ we have that $[(v, v'), (w, w')]$ is an edge if and only if $v = w$ or $[v, w] \in E$, and either $v' = w'$ or $[v', w'] \in E'$.
- **Lexiographic Product:** In the lexiographic product $G \bullet G'$ we have that $[(v, v'), (w, w')]$ is an edge if and only if either $[v, w] \in E$, or $v = w$ and $[v', w'] \in E'$.

Definitions 2. *A graph is dense if $|E|$ is "close" to $|V|^2$ and sparse if $|E|$ is "much less" than $|V|^2$ [8].*

10.2 Ring Theory Background

Definitions 3. *A ring is a set of elements R with operations $+, \cdot$ such that, if we have arbitrary $r_1, r_2, r_3 \in R$ then:*

- *There is a zero element: There is $0 \in R$ such that $r_1 + 0 = r_1 = 0 + r_1$*
- *The ring is closed and associative under multiplication and addition: We have $r_1 + r_2 = r_2 + r_1, r_1 \cdot r_2 = r_2 \cdot r_1 \in R$*
- *There are an additive inverses: There is $-r_1 \in R$ such that $r_1 + -r_1 = 0$*

- *Multiplication distributes over addition: We have* $r_1(r_2+r_3) = r_1r_2 + r_1r_3$ *and* $(r_1 + r_2)r_3 = r_1r_3 + r_2r_3$.

  We will focus on commutative rings with identity, that is rings which are

- *commutative:* $r_1r_2 = r_2r_1$,

- *and have an identity: there is 1* $\in R$ *such that* $r_1 \cdot 1 = 1 \cdot r_1$.

Such rings are also called commutative rings with unity.

An ideal I *is a subgroup of* R *which is an additive group such that* $AI \subseteq I$. *That is* $r \in R$ *and* $i \in I$ *implies* $ri \in I$. *(Recall that an additive group is a set of elements with operation + such that there is a zero element, every element has an inverse, the operation is associative, and if* a *and* b *are in the set, then* $a+b$ *is in the set as well).*

An ideal I *is proper if* $I \neq R$.

The principal ideal, denoted Ra *or* (a), *for* $a \in R$, *is the set* $\{ra : r \in R\}$.

An ideal P *is prime if* $P \neq R$, *and if* $r_1r_2 \in P$ *implies* $r_1 \in P$ *or* $r_2 \in P$.

An ideal M *is maximal if* $M \neq R$ *and if, given for an ideal* I *that* $M \subseteq I \subseteq R$, *then* $M = I$ *or* $I = R$.

A ring is semilocal if it contains finitely many maximal ideals. The set of maximal ideals is denoted Max(R).

The Jacobson radical $J(R)$ *is equal to the intersection of all of the maximal ideals.*

A ring R *is local if it contains a unique maximal ideal* M. *Such a ring is often denoted* (R, M).

A ring is Noetherian if it satisfies one of the following equivalent conditions:

1. *Every non-empty set of ideals in* R *has a maximal element.*

2. *If we have a sequence of ideals* I_1, I_2,, I_3... *such that* $I_i \subseteq I_{i+1}$ *for all* $i \in \mathbb{N}$, *then there is* n *such that* I_n=I_{n+1}=....

3. *Every ideal in* A *is finitely generated.*

A ring is Artin if for any sequence of ideals I_1, I_2,, I_3... *such that* $I_i \supseteq I_{i+1}$ *for all* $i \in \mathbb{N}$, *there is* n *such that* I_n=I_{n+1}=....

An element x *of a ring is an idempotent if* $x^2 = x$.

A unit in a ring is any element with a multiplicative inverse.

Theorem 10.2.1. *If the ring is semilocal then* $J(R)$ *is equal to the product of maximal ideals.*

Theorem 10.2.2. *Every ideal* $I \neq R$ *is contained in a maximal ideal.*

Theorem 10.2.3. *Any Artin ring is also Noetherian.*

Theorem 10.2.4. *Every Artin ring is equal to the finite direct product of Artin local rings, up to isomorphism.*

10.3 Lattices

There are two equivalent definitions for a lattice:

Definition 10.3.1. *A lattice L is an algebra with two binary operations ($\wedge$ and $\vee$) satisfying for all a, b, c in L the following conditions:*

- *For all a, b, there is a unique $a \wedge b \in L$.*
- *For all a, b, there is a unique $a \vee b \in L$.*
- $a \vee b = b \vee a$
- $a \wedge b = b \wedge a$
- $a \wedge (b \wedge c) = (a \wedge b) \wedge c$
- $a \vee (b \vee c) = (a \vee b) \vee c$
- $a \wedge (b \vee c) = a$
- $a \vee (a \wedge b) = a$.

Note that $\vee$ can also be denoted $+$ and $\wedge$ can also be denoted $\cdot$ We say $a \geq b$ if $a \vee b = a$.

Definition 10.3.2. *A lattice is a partially ordered set in which every pair of elements a, b has a greatest lower bound (represented by $a \wedge b$ or glb(a,b)) and a least upper bound (represented by $a \vee b$ or lub(a,b))within the set [6].*

Definitions 4. *A lattice is modular if the following postulate is met: for elements a, b, c of a lattice, $a \geq b$ implies $a \vee (b \vee c) = (a \wedge b) \vee (a \wedge c) = b \vee (a \wedge c)$.*

A lattice is distributive if the following postulate is met: for any elements a, b, c in the lattice $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ [6].

A lattice is complete if for any arbitrary set of elements there is a meet and a join [7].

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