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Enumeration of $\mathcal{A}^{\lambda}$-Ice models and Strict Gelfand-Tsetlin Patterns

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Abstract. We want to generalize the alternating sign matrix conjecture. We identify the 1-to-1 correspondence between type $\mathcal{A}^\lambda$ ice models, whose boundary conditions are determined by integer partitions, and strict Gelfand-Tsetlin patterns. We use these connections to derive a recursive relationships on the enumeration of ice models determined by integer partitions.

Contents

The fact that a conjecture resists vigorous attacks by skilled practitioners in an impetus for us either to sharpen out existing tools, or else create new ones. The value of a proof of an outstanding conjecture should be judged, not by its cleverness and elegance, and not even by its 'explanatory power,' but by the extent in which it enlarges our toolbox.

Doron Zeilberger in *Introduction to proof of the alternating sign matrix conjecture* ([**12**])

Acknowledgments

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Preface

In Chapter 1, We introduce alternating sign matrices (ASM) and the ASM conjecture. Chapter 2 is an exposition on Kuperberg's approach to prove the ASM conjecture. In Chapter 3, we briefly introduce Gelfand-Tsetline patterns, which are important for enumerating type $\mathcal{A}^\lambda$ ice. Type $\mathcal{A}^\lambda$ ice is the main topic of this thesis, and its properties are discussed in Chapter 4. Chapter 5 shows rank 2&3 $\mathcal{A}^\lambda$ ice models, which are the basic cases for enumeration; in Chapter 6, we provide an recursive formula that generalizes the enumeration to any rank. Connection to representation theory is presented in Chapter 7.

CHAPTER 1

Introduction

Alternating sign matrices generalizes permutation matrices[1].

DEFINITION 1. An alternating-sign matrix (or ASM) is a matrix with entries 1, 0, and -1, such that

(1) the non-zero entries alternate in sign in each row and column;
(2) the sum of entries in each row and each column is 1.

Let A_n denote the number of $n \times n$ ASMs. The sequence A_n is

$$(1, 2, 7, 42, 429, 7436, 218348, 10850216, 911835460, \cdots)$$

and we can see that this sequence grows faster than $n!$.

For example $A_3 = 7$ and the matrices are :

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$
$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$
$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \text{and} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

It follows from the definition that:

(1) each alternating sign matrix can have only 1 in its top row;
(2) the mirror image of an $n \times n$ alternating sign matrix is still an ASM
(3) if we rotate an $n \times n$ alternating sign sign matrix clockwise by 90°, we still have an ASM.

We can divide the $n \times n$ alternating sign matrices into n subsets according to where 1 is in the first row. Define $A_{n,k}$ to be the number of $n \times n$ alternating sign matrices with $(1, k)$-th entry being 1. In the example above, we have $A_{3,1} = 2$, $A_{3,2} = 3$ and $A_{3,3} = 2$.

[1]The presentation in Chapter 1 follows from David M. Bressoud's *Proofs and Confirmations: the story of the alternating sign matrix conjecture*([**2**]).

We can organize the $A_{n,k}$ information into a triangular array with the kth entry in the nth row is $A_{n,k}$:

$$\begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & & & & 1 & & 1 & & & & \\ & & & 2 & & 3 & & 2 & & & \\ & & 7 & & 14 & & 14 & & 7 & & \\ & 42 & & 105 & & 135 & & 105 & & 42 & \\ 429 & & 1287 & & 2002 & & 2002 & & 1287 & & 429. \end{array}$$

Note that $A_{n,k} = A_{n,n-k}$ because the mirror image of an alternating matrix with a 1 at the kth entry on the top row must be an alternating matrix with a 1 at the $n-k$th entry on the top row. In addition, notice that

$$A_{n,1} = A_{n,n} = A_{n-1}. \tag{1}$$

By investigating the patterns in this Pascal triangle, Mills, Robbins, and Rumsey proposed the refined ASM conjecture in [**9**].

Theorem 0.1 (The refined ASM conjecture). *For* $1 \leq k < n$*,*

$$\frac{A_{n,k}}{A_{n,k+1}} = \frac{\binom{n-2}{k-1} + \binom{n-1}{k-1}}{\binom{n-2}{n-k-1} + \binom{n-1}{n-k-1}} = \frac{k(2n-k-1)}{(n-k)(n+k-1)}. \tag{2}$$

The refined ASM conjecture is equivalent to the following claim, which gives an explicit formula for $A_{n,k}$:

Theorem 0.2. *For* $1 \leq k \leq n$*,*

$$A_{n,k} = \binom{n+k-2}{k-1} \frac{(2n-k-1)!}{(n-k)!} \prod_{j=0}^{n-2} \frac{(3j+1)!}{(n+j)!}. \tag{3}$$

Theorem 0.2 combined with Equation 1 implies

$$\begin{aligned} A_n &= A_{n+1,1} \\ &= \binom{n}{0} \frac{(2n)!}{(n)!} \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+1+j)!} \\ &= \frac{(2n)!}{n!} \prod_{j=0}^{n-1} \frac{1}{n+1+j} \frac{(3j+1)!}{(n+j)!} \\ &= \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}, \end{aligned}$$

which is the ASM conjecture.

THEOREM 0.3 (The ASM Conjecture). *There are*

$$A_n = \frac{1!4!7!\cdots(3n-2)!}{n!(n+1)!(n+2)!\cdots(2n-1)!} \tag{4}$$

$n \times n$ ASMs.

The ASM conjecture was first proved by Doron Zeilberg in 1992 in [**12**], and Gerg Kuperberg gave another shorter proof of Theorem 0.3 in 1996 in [**6**]. Kuperberg's techniques use the bijection between $n \times n$ alternating sign matrices and $n \times n$ square ice models with domain wall boundary conditions and Yang-Baxter equation from statistical mechanics. We will discuss Kuperberg's method in Chapter 2. Subsequently, in [**13**] Zeilberger proved the refined ASM conjecture using Kuperberg's method together with techniques from q-calculus and orthogonal polynomials.

For completeness, we include a proof of the equivalence between Theorem 0.1 and Theorem 0.2 here.

PROPOSITION 0.4. *For $1 \leq k \leq n$, we have*

$$A_{n,k} = \binom{n+k-2}{k-1}\frac{(2n-k-1)!}{(n-k)!}\prod_{j=0}^{n-2}\frac{(3j+1)!}{(n+j)!}$$

if and only if

$$\frac{A_{n,k}}{A_{n,k+1}} = \frac{\binom{n-2}{k-1}+\binom{n-1}{k-1}}{\binom{n-2}{n-k-1}+\binom{n-1}{n-k-1}} = \frac{k(2n-k-1)}{(n-k)(n+k-1)}.$$

PROOF. To prove the forward direction, notice that

$$\begin{aligned}\frac{A_{n,k}}{A_{n,k+1}} &= \frac{\binom{n+k-2}{k-1}\frac{(2n-k-1)!}{(n-k)!}\prod_{j=0}^{n-2}\frac{(3j+1)!}{(n+j)!}}{\binom{n+k-1}{k}\frac{(2n-k-2)!}{(n-k+1)!}\prod_{j=0}^{n-2}\frac{(3j+1)!}{(n+j)!}}\\ &= \frac{\frac{(n+k-2)!}{(k-1)!(n-1)!}\frac{(2n-k-1)!}{(n-k)!}}{\frac{(n+k-1)!}{(k)!(n-1)!}\frac{(2n-k-2)!}{(n-k+1)!}}\\ &= \frac{k(2n-k-1)}{(n+k-1)(n-k)},\end{aligned}$$

which is what Theorem 0.1 states.

Now suppose

$$\frac{A_{n,k}}{A_{n,k+1}} = \frac{\binom{n-2}{k-1}+\binom{n-1}{k-1}}{\binom{n-2}{n-k-1}+\binom{n-1}{n-k-1}} = \frac{k(2n-k-1)}{(n-k)(n+k-1)},$$

then we must have

$$A_{n,k} = A_{n,1} \times \prod_{j=1}^{k-1}\frac{(n-j)(n+j-1)}{(j)(2n-j-1)}$$

$$= A_{n,1}\frac{(n+k-2)!(2n-k-1)!}{(n-k)!(k-1)!(2n-2)!}.$$

Since $A_{n+1,1} = A_n = \sum_{k=1}^{n} A_{n,k}$, and then

$$A_n = A_{n,1}\sum_{k=1}^{n}\frac{(n+k-2)!(2n-k-1)!}{(n-k)!(k-1)!(2n-2)!} = A_{n-1}\sum_{k=1}^{n}\frac{(n+k-2)!(2n-k-1)!}{(n-k)!(k-1)!(2n-2)!}.$$

Lemma 0.5. *Let a, b, m be positive integers, then*

$$\binom{a+b+m+1}{m} = \sum_{k=0}^{m}\binom{a+k}{k}\binom{b+m-k}{m-k}.$$

Proof. By the generalized binomial theorem, we have

$$(1-x)^{-s} = \sum_{k=0}^{\infty}\binom{s+k-1}{k}x^k,$$

then

$$\begin{aligned}\sum_{m=0}^{\infty}\binom{a+b+1+m}{m}x^m &= (1-x)^{-a-b-2}\\ &= (1-x)^{-a-1}(1-x)^{-b-1}\\ &= \sum_{k=0}^{\infty}\binom{a+k}{k}x^k\sum_{j=0}^{\infty}\binom{b+j}{j}x^j.\end{aligned}$$

If we match the coefficient of x^m on both sides, we would get

$$\binom{a+b+m+1}{m} = \sum_{k=0}^{m}\binom{a+k}{k}\binom{b+m-k}{m-k}.$$

□

With Lemma 0.5, we must have

$$\begin{aligned}A_n &= A_{n-1}\sum_{k=1}^{n}\frac{(n+k-2)!(2n-k-1)!}{(n-k)!(k-1)!(2n-2)!}\\ &= A_{n-1}\frac{(n-1)!(n-1)!}{(2n-2)!}\sum_{k=1}^{n}\binom{n+k-2}{n-1}\binom{2n-k-1}{n-1}\\ &= A_{n-1}\frac{(n-1)!(n-1)!}{(2n-2)!}\binom{3n-2}{2n-1}\\ &= \frac{(n-1)!(3n-2)!}{(2n-2)!(2n-1)!}A_{n-1}\end{aligned}$$

$$= \prod_{k=2}^{n} \frac{(k-1)!(3k-2)!}{(2k-2)!(2k-1)!}.$$

LEMMA 0.6.

$$\prod_{k=2}^{n} \frac{(k-1)!(3k-2)!}{(2k-2)!(2k-1)!} = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!} \tag{5}$$

PROOF. Proceed by induction on n. It is obvious that when $n = 2$, we have

$$\prod_{k=2}^{2} \frac{(k-1)!(3k-2)!}{(2k-2)!(2k-1)!} = \frac{1!4!}{2!3!} = 2 = \frac{1!}{2!}\frac{4!}{3!} = \prod_{j=0}^{2-1} \frac{(3j+1)!}{(2+j)!}.$$

Now suppose that Equation 5 is true for some $m \in \mathbb{N}$, then

$$\begin{aligned}
\prod_{k=2}^{m+1} \frac{(k-1)!(3k-2)!}{(2k-2)!(2k-1)!} &= \frac{m!(3m+1)!}{(2m)!(2m+1)!} \prod_{k=2}^{m} \frac{(k-1)!(3k-2)!}{(2k-2)!(2k-1)!} \\
&= \frac{m!(3m+1)!}{(2m)!(2m+1)!} \prod_{j=0}^{m-1} \frac{(3j+1)!}{(m+j)!} \\
&= \frac{m!(3m+1)!(m+1)(m+2)\cdots(2m)}{(2m)!(2m+1)!} \prod_{j=0}^{m-1} \frac{(3j+1)!}{(m+1+j)(m+j)!} \\
&= \frac{(3m+1)!}{(2m+1)!} \prod_{j=0}^{m-1} \frac{(3j+1)!}{(m+1+j)!} \\
&= \prod_{j=0}^{m} \frac{(3j+1)!}{(m+1+j)!}.
\end{aligned}$$

Therefore Equation 5 is true for all $n \in \mathbb{N}$. □

Now we have

$$\begin{aligned}
A_{n,k} &= A_{n,1} \frac{(n+k-2)!(2n-k-1)!}{(n-k)!(k-1)!(2n-2)!} \\
&= \frac{(n+k-2)!(2n-k-1)!}{(n-k)!(k-1)!(2n-2)!} \prod_{j=0}^{n-2} \frac{(3j+1)!}{(n-1+j)!} \\
&= \binom{n+k-2}{k-1} \frac{(2n-k-1)!}{(n-k)!} \prod_{j=0}^{n-2} \frac{(3j+1)!}{(n+j)!}.
\end{aligned}$$

□

CHAPTER 2

Kuperberg

1. Partition Function

The presentation about partition functions comes from Schroder's *Introduction to Thermal Physics* ([**11**]).

DEFINITION 2. Suppose we have a system of interacting particles, in thermal contact with a reservoir at temperature T. If s is a particular configuration/state of the system, let $E(s)$ be the system's energy in that state. Then a partition function is

$$Z = \sum_s e^{-E(s)/kT} \tag{6}$$

where $k = 1.38 \times 10^{-23}\text{m}^2\text{kg/s}^2\text{K}$ is the Boltmann constant. Let $\beta = \frac{1}{kT}$, then

$$Z = \sum_s e^{-\beta E(s)}. \tag{7}$$

The partition function represents a sum over all possible states of the system.

Partition functions are central to statistical mechanics, because for a thermodynamically large system, a partition function is like the root of a tree from which we can derive the total energy, free energy, entropy and pressure of the system. For example, the probability that the system occupies a certain state s is

$$P(s) = \frac{1}{Z} e^{-\beta E(s)}.$$

We can also derive that the average energy of the system is

$$\overline{E} = -\frac{\partial \ln Z}{\partial \beta}.$$

Although partition functions are powerful, finding an explicit formula of Z is difficult. So far, we only have exact solutions to a limited set of models, among which is the square ice.

2. Square Ice

DEFINITION 3 (Six-Vertex Model, Square Ice). Let $\mathcal{G}$ be a directed graph, such that every internal vertex has four neighbors and each boundary vertex has only one neighbor. An ice state (also called a six-vertex state) of $\mathcal{G}$ is a state such that each vertex has in-degree 2 and out-degree 2. The six-vertex model (or ice model) refers the set of all ice states. If $\mathcal{G}$ is on a square grid, the set of ice states is called square ice.

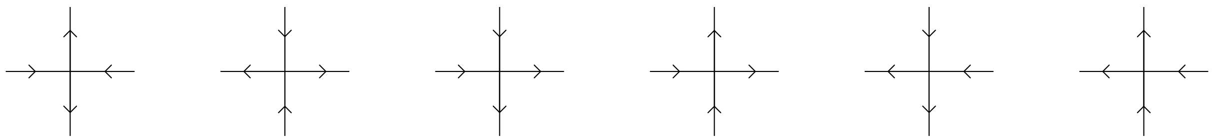


FIGURE 1. Six permissible vertex configurations.

It is easy to see that there are six valid states for each vertex as shown in Figure 1. The energy of a state is a function of the states of each vertex. Define the total energy E to be

$$E = \sum_x n_x \epsilon_x$$

where $x \in \{\mathrm{EW}, \mathrm{NS}, \mathrm{NW}, \mathrm{SW}, \mathrm{SW}, \mathrm{NE}, \mathrm{SE}\}$ and n_x denote the number of vertices in state x, and ϵ_x is the energy associated with the vertex configuration x. The partition function Z of a square ice model should be

$$Z = \sum_s e^{-\beta E(s)}.$$

Note that if we set

$$\epsilon_{\mathrm{EW}} = \epsilon_{\mathrm{NS}} = \epsilon_{\mathrm{NW}} = \epsilon_{\mathrm{SW}} = \epsilon_{\mathrm{NE}} = \epsilon_{\mathrm{SE}} = 0,$$

then the partition function Z equals the total number of valid states, which is the cardinality of the ice model.

In 1967, Elliott H. Lieb gave the exact solution to square ice in [**8**], when the number of vertices approaches infinity. The solution for the square ice model when $N \to \infty$ is

$$Z = \left(\frac{4}{3}\right)^{\frac{3N}{2}} \approx 1.5396^N,$$

and $\left(\frac{4}{3}\right)^{\frac{3}{2}}$ is called the Lieb's square ice constant. In 1993, Korepin, Bogoliubov and Izergin gave the formula for $n \times n$ square ice in [**5**].

3. Domain Wall Boundary Condition and Bijection with ASMs

DEFINITION 4. Let G be a tetravalent graph in a six-vertex state. Restrictions on the orientation of the boundary edges are called boundary conditions. A domain wall boundary condition is defined as edges pointing inward at the sides and outward at the top and bottom.

A square ice state with domain wall boundary condition can be converted to an ASM by the correspondence in Figure 2.

PROPOSITION 3.1. *The set of $n \times n$ square ice with domain wall boundary condition is in bijection with $n \times n$ alternating sign matrices.*

With the proposition, to count the number of $n \times n$ of alternating sign matrices becomes the same as finding the partition function Z where we set the energy of every state to be 0.

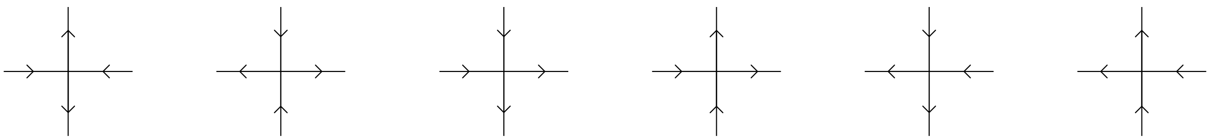


FIGURE 2. Six permissible vertex configurations and their corresponding matrix entry.

4. Boltzmann Weights, Spectral Parameter and State Sums

Let s be an ice state with domain wall boundary conditions. In state $\mathcal{G}$, each vertex v is given a weight $w(s,v)$. The weight of the state is $W(s) = \prod_v w(s,v)$. In statistical mechanics, we call multiplicative weights *Boltzmann weights*.

Note that $E(s) = 0$ for every state means $e^{-\beta E(s)} = 1$, and is the same as defining $W(s) = 1$ for every state. Under this context, the partition function

$$Z = \sum_s W(s) \tag{8}$$

is often called a state sum, as it is the sum of weights of all states. So the enumeration of ASM's is also equal to a six-vertex state sum in which all weights are 1.

In addition, Mills, Robins and Rumsey suggest using another set of weights and that leads to x-enumeration.

DEFINITION 5 (Generating function, in [**9**]). Let us denote $A_n(x)$ the generating function for the set of all $n \times n$ alternating sign matrices. That is $A_n(x) \in \mathbb{Z}[x]$ such that the coefficient of x^m is the number of $n \times n$ alternating sign matrices with m entries that are equal to -1.

Mills, Robbins and Rumsey provided the first few functions:

$$\begin{aligned} A_1(x) &= 1, \\ A_2(x) &= 2, \\ A_3(x) &= 6 + x \\ A_4(x) &= 24 + 16x + 2x^2. \end{aligned}$$

The generating functions lead to Kuperberg's x-enumerations of alternating sign matrices in [**6**].

DEFINITION 6 (x-enumeration, in [**6**]). If x is a number, define the x-enumeration $A(n;x)$ of $n \times n$ alternating sign matrices as their total weight, where the weight of an individual matrix is x^k if it has k entries equal to -1.

As we can see, the x-enumeration is more general then the ASM conjecture, which is the 1-enumeration. The 2-enumeration

$$A_{n;2} = 2^{n(n-1)/2}$$

is proved by Elkies, Kuperberg and Larsen in [**4**]. The 3-enumeration was conjectured by Mills, Robbins and Rumsey, and proved by Kuperberg in [**6**]:

Theorem 4.1. *ASM's are* 3 *enumerated by*

$$A(2n+1;3) = \left(3^{n(n+1)/2}\frac{2!5!8!\cdots(3n-1)!}{(n+1)!(n+2)!\cdots(2n)!}\right)^2$$

$$A(2n;3) = 3^{n-1}\frac{(3n-1)!(n-1)!}{(2n-1)!}A(2n-1;3).$$

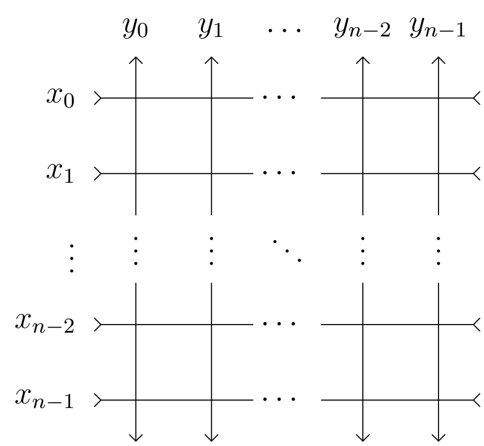
As we can see, we need a weight system that connects the state sum and the generating function. To be consistent with the notation that Kuperberg used in [**7**] and Bressould's notation in [**2**], we adopt the following abbreviations:

$$\overline{x} = x^{-1}$$

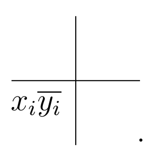
$$\sigma(x) = x - \overline{x}$$

$$[x] = \frac{x-\overline{x}}{a-\overline{a}} = \sigma(x)\overline{\sigma(a)}$$

Let $\mathcal{G}$ be an $n \times n$ graph with domain wall boundary conditions. We can assign arbitrary parameters $\vec{x} = x_0, x_1, \cdots, x_{n-1}$ and $\vec{y} = y_0, y_1, \cdots, y_{n-1}$ to the horizontal and vertical lines.



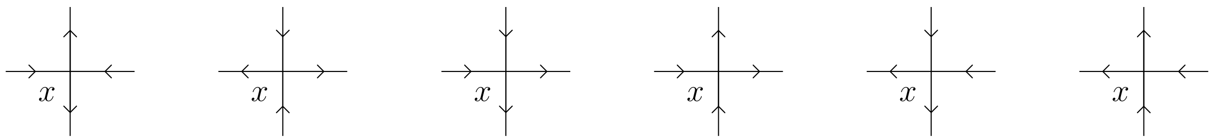
We call $\vec{x}$ and $\vec{y}$ *spectral parameters*. A vertex at the intersection of the lines with spectral parameters x_i and y_i is given the spectral parameter $x_i\overline{y_i}$, and for convenience we label the vertex as



In [**6**] and [**7**], Kuperberg assigned weights, called R-matrix, to each of the 6 admissible vertex configurations as shown in Figure 3.

Definition 7 (Partition function as state sum). Let $\mathcal{M}$ be an $n \times n$ ice model. If we associate the ice model with spectral parameter $\vec{x}$ and $\vec{y}$, and R-matrix as weights, then the partition function for $\mathcal{M}$ is the resulting state sum $Z(n;\vec{x},\vec{y})$.

The partition function have been found by Korepin, Bogoliubov, and Izergin in [**5**].



EW	NS	NW	SW	NE	SE
1	-1	0	0	0	0
$\sigma(a^2)$	$\sigma(\alpha^2)$	$\sigma(ax)$	$\sigma(a\overline{x})$	$\sigma(a\overline{x})$	$\sigma(ax)$

FIGURE 3. Name, equivalent matrix entry, and eight of the 6 permissible vertex configurations. (Note: x is the spectral parameter of each configuration.)

THEOREM 4.2 (Korepin, Bogoliubov, and Izergin).

$$Z(n;\vec{x},\vec{y}) = \frac{\prod_{i=1}^{n} x_i\overline{y_i} \prod_{1\le i,j\le n} [x_i\overline{y_j}][ax_i\overline{y_j}]}{\prod_{1\le i\le j\le n} [x_i\overline{x_j}][y_i\overline{y_j}]} \det M \tag{9}$$

where M is the $n \times n$ matrix with entries

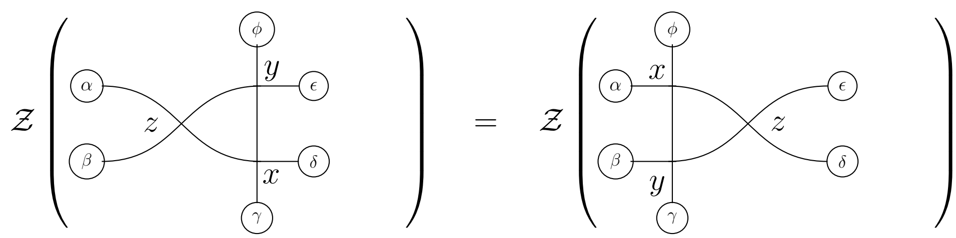
$$M_{ij} = \frac{1}{[x_i\overline{y_j}][ax_i\overline{y_j}]}.$$

To prove Theorem 4.2, one needs the Yang-Baxter equation.

5. Yang-Baxter Equation

Success of Kuperberg's x-enumeration depends on the properties of the weight assignment R-matrix, and the key property is the Yang-Baxter equation from [**1**].

THEOREM 5.1 (Yang-Baxer equation). *Given a triangle within a six-vertex lattice and labels x, y and z such that $xyz = \overline{a}$, the following identity of partition functions holds:*

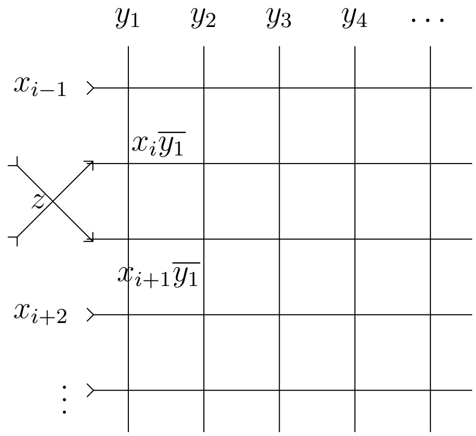


(10)

for any choice of boundary arrow decorations $\alpha, \beta, \gamma, \delta, \epsilon, \phi$.

Yang-Baxter equation suggests that summing over the weight of all possible ways of filling ou the twist diagram on the left will give the same result as summing over the weight of all possible ways of filing out the twist diagram on the right. These equation consists 64 numerical equalities, because there are $2^6 = 64$ ways to orient the 6 boundary arrows $\alpha, \beta, \gamma, \delta, \epsilon, \phi$. However, in order to have a six-vertex lattice, we must have 3 in-arrows and 3 out-arrows, and this leaves us only 20 non-zeros equations. In [**7**], Kuperberg also points out that the equation has 3 symmetries. A detailed proof can be found on page 235-237 in [**2**].

Given a directed graph, we can attach a new vertex in the southwest configuration, rotated 45° closewise, along the left edge between the ith and $i+1$st rows. Adding the new vertex



introduces a triangle into the lattice, and if we set $xyz = \overline{a}$, then the effect of adding the new vertex is to multiply the weight of the lattice by $\sigma(a\overline{z})$, or equivalently to multiply the partition function by $\sigma(a\overline{z})$. Yang-Baxter says that we can move the vertex all the way to the right of the lattice along the horizontal direction and interchange the labels on line i and $i+1$ without changing the partition function. Note that after moving the new vertex to the right, the weight of this vertex is still $\sigma(a\overline{z})$. So we can interchange the labels on rows i and $i+1$ without changing the partition function. A similar argument we can do this to adjacent columns too. Therefore the partition function has symmetry.

PROPOSITION 5.2 (Baxter). *The function $Z(n; \vec{x}, \vec{y})$ is symmetric in the x_i's and in the y_i's.*

6. Connecting Partition Functions and x-enumerations

Kuperberg saw how to connect partition functions and x-enumeration. Let $\vec{1} = (1, 1, \cdots, 1)$ and $x = a + 2 + \overline{a}$. The x-enumeration that Kuperberg proved in [**7**] was

$$A(n; x) = \frac{Z(n; \vec{1}, \vec{1})}{\sigma(a)^{n^2-n}\sigma(a^2)^n}. \tag{11}$$

If $a = \omega = e^{2\pi i/3}$, then $x = 1$, $\sigma(a) = 1$, and we have

$$A_n = A(n; 1) = Z(n, \vec{1}, \vec{1}).$$

In addition to 2 & 3-enumeration of ASMs in [**4, 6**], Kuperberg also proved weighted enumeration for symmetry classes of ASMs in [**7**].

CHAPTER 3

Gelfand-Tsetlin pattern

As Mills, Robbins and Rumsey pointed out in [**9**], alternating sign matrices are related to combinatorial objects called monotone triangles. The monotone triangles show up as Gelfand-Tsetlin pattern in representation theory.

1. Gelfand-Tsetlin pattern and monotone triangles

For each $n \in \mathbb{N}$, let X_n be the set of all triangular arrays $(x_{i,j})_{1\le i\le j\le n}$ with $x_{i,j} \in \mathbb{R}$. Then X_n inherits a vector space structure under the obvious isomorphism $X_n \simeq \mathbb{R}^{n(n+1)/2}$.

Definition 8 (Gelfand-Tsetline pattern and monotone triangle). The Gelfand-Tsetlin pattern is a triangular array

$$
\begin{array}{ccccccccccccccc}
x_{1,1} & & x_{1,2} & & x_{1,3} & & x_{1,4} & & x_{1,5} & & \cdots & & x_{1,n} \\
& x_{2,2} & & x_{2,3} & & x_{2,4} & & x_{2,6} & & \cdots & & x_{2,n} & \\
& & x_{3,3} & & x_{3,4} & & x_{3,5} & & \cdots & & x_{3,n} & & \\
& & & x_{4,4} & & x_{4,5} & & \cdots & & x_{4,n} & & & \\
& & & & \ddots & & \cdots & & \iddots & & & & \\
& & & & & x_{n-1,n-1} & & x_{n-1,n} & & & & & \\
& & & & & & x_{n,n} & & & & & &
\end{array}
$$

$(x_{i,j})_{1\le i\le j\le n} \in X_n$ satisfying the following inequalities:

(1) $x_{i,j} \ge 0$ for $1 \le i \le j \le n$; and
(2) $x_{i,j} \ge x_{i+1,j+1} \ge x_{i,j+1}$ for $1 \le i \le j \le n-1$.

A monotone triangle is a Gelfand-Tsetlin pattern whose entries are all positive integers. In this paper, we will refer to monotone triangles as Gelfand-Tsetline patterns.

In this definition, the inequalities state that each entry is positive, and each entry is weakly greater than its bottom-right neighbor and weakly less than its bottom-left neighbor. As an example, the followings are Gelfand Tsetlin patterns:

$$
\begin{array}{ccccccc}
7 & & 5 & & 3 & & 2 \\
& 6 & & 5 & & 3 & \\
& & 4 & & 3 & & \\
& & & 3 & & &
\end{array}
\quad , \quad
\begin{array}{ccccccc}
7 & & 5 & & 3 & & 2 \\
& 6 & & 4 & & 2 & \\
& & 4 & & 4 & & \\
& & & 4 & & &
\end{array}
\quad \text{and} \quad
\begin{array}{ccccccccc}
8 & & 6 & & 4 & & 2 & & 1 \\
& 7 & & 5 & & 4 & & 2 & \\
& & 6 & & 4 & & 4 & & \\
& & & 4 & & 4 & & & \\
& & & & 4 & & & &
\end{array}
\quad .
$$

Note that in a Gelfand-Tsetlin pattern $(x_{i,j}) \in X_n$, it is possible to have

$$x_{i,j} = x_{i+1,j} = x_{i+1,j+1} = x_{i+2,j+1}.$$

If this happens, we will say that X_n has a *collision* at (i, j). Note that all collisions look like parallelograms in the triangular array. In our example above, the second Gelfand Tsetlin pattern has a collision at $(2, 3)$ and the third pattern has a collision at $(2, 4)$ and a collision at $(3, 4)$.

If a Gelfand-Tsetlin pattern is collision free, then we call it a strict Gelfand-Tsetlin pattern:

Definition 9. Let $(x_{i,j}) \in X_n$ be a Gelfand-Tsetline pattern. We call X_n a strict Gelfand-Tsetlin pattern if $x_{i,j} > x_{i,j+1}$ for all $1 \le i \le j \le n-1$.

In a strict Gelfand-Tsetlin pattern, every row is strictly decreasing.

Given a $n \times n$ ASM, we can produce a strict Gelfand Tsetlin pattern through the following algorithm:

(1) Replace every entry in the matrix by the sum of all elements in the same column below it, including the entry itself. We should get a $n \times n$ matrix with entries 0 and 1.
(2) In every row, record the columns containing a 1 and put it into the strict Gelfand Tsetline pattern.

For example, given the 5×5 ASM $\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$, after step (1) in the above algorithm, we get the matrix $\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$, and step (2) would give us the strict Gelfand-Tsetlin pattern

$$\begin{array}{ccccccccc} 5 & & 4 & & 3 & & 2 & & 1 \\ & 5 & & 3 & & 2 & & 1 & \\ & & 4 & & 3 & & 1 & & \\ & & & 3 & & 2 & & & \\ & & & & 2 & & & & \end{array} .$$

The algorithm above is invertible. So there is a bijection between $n \times n$ ASMs and strict Gelfand-Tsetlin pattern with n rows, and the first row is the integer partition $(n, n-1, n-2, \cdots, 2, 1)$, which are sometimes referred to as *complete monotone triangles.*

2. Ice Models and Gelfand-Tsetlin Pattern

Strict Gelfand-Tsetlin patterns are also in bijection with ice models by the following algorithm:

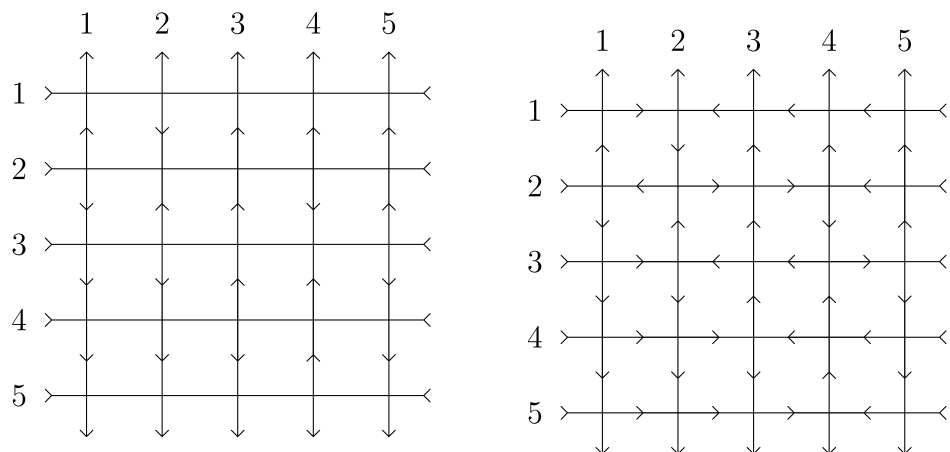
Suppose (x_{ij}) is an strict Gelfand-Tsetlin pattern with n rows, and the first row is the integer partition $(n, n-1, n-2, \cdots, 2, 1)$. Let $\lambda_i = \{x_{i,i}, x_{i,i+1}, \cdots, x_{i,n}\}$, which is an integer partition. The pattern produces the following rectangular ice:

(1) Draw an ice lattice with n rows and $n = x_{1,1}$ columns. Let arrows on the left and right boundaries point into the ice and arrows along the bottom boundary point out of the ice.
(2) For every $1 \leq i \leq n$ and $1 \leq j \leq x_{1,1}$. In the ice, the arrow above the vertex (i, j) points up if $j \in \lambda_i$.
(3) All the other arrows along the vertical direction points down.
(4) Starting at the top-right vertex in the lattice and progressing in an S-shape, fill in all the horizontal arrows in the ice model diagram.

In our running example, consider the GT pattern

$$\begin{array}{ccccccccc} 5 & & 4 & & 3 & & 2 & & 1 \\ & 5 & & 3 & & 2 & & 1 & \\ & & 4 & & 3 & & 1 & & \\ & & & 3 & & 2 & & & \\ & & & & 2 & & & & \end{array}$$

. The procedure to produce an ice model is illustrated in Figure 1.



(A) Step 2& 3 determines all vertical arrows

(B) Step 4 is to fix all horizontal arrows

FIGURE 1. The procedures of producing an ice model from a GT pattern.

It is easy to see that this construction is bijective.

CHAPTER 4

Type $\mathcal{A}^{\lambda}$ Ice

In this paper, we discuss ice models which boundary condition is determined by an integer partition λ. This boundary condition is the same as the domain wall boundary condition except along the top edge, and the partition function of these models has been studied by Bruaker, Bump and Friedberg in [**3**].

1. Boundary Condition Defined by Integer Partition λ

Definition 10. Let $\lambda = (x_1, x_2, \cdots, x_{n-1}, x_n)$ be an integer partition. (Usually we set $x_n = 1$.) The partition λ defines boundary condition for an $n \times x_n$ ice model such that along the top, arrows point outward only for those columns that correspond to parts of the integer partition λ, and all other boundaries conditions are the same as in the domain wall boundary condition.

Let $|\mathcal{A}^{\lambda}|$ denote the number of ice models with boundary condition determined by λ.

As an example, when $\lambda = (6, 3, 1)$, the ice model looks like:

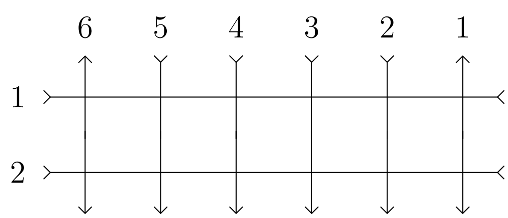
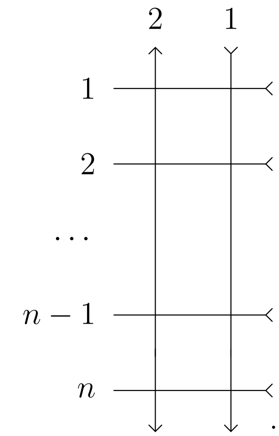


Figure 1. Boundary condition of $\mathfrak{A}^{[6,3,1]}$.

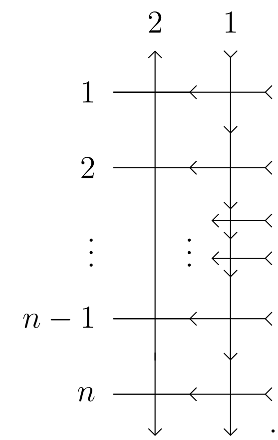
2. Properties of the $\mathcal{A}^{\lambda}$ Ice Models

Lemma 2.1 (Shift Invariant). *For any integer partition λ and $c \in \mathbb{N}$, we must have $|\mathcal{A}^{\lambda}| = |\mathcal{A}^{\lambda+c}|$, where $\lambda + c$ is the integer partition resulted from adding a constant c to every entry in λ.*

Proof. Let $\lambda = (x_1, x_2, \cdots, x_n)$ where $x_n = 1$, and suppose $c > 0$. Then $\lambda + c = (x_1 + c, x_2 + c, \cdots, x_n + c)$. We will show the above statement is true for $c = 1$. When $c = 1$, we add a new column to the right of the ice model and column 1 and 2 of the new ice model must look like



But there is only one way to fill in this part of the ice lattice, namely



Therefore shifting the entire ice model to the left does not add any new permissible states. So $|\mathcal{A}^{\lambda}| = |\mathcal{A}^{\lambda+c}|$. $\square$

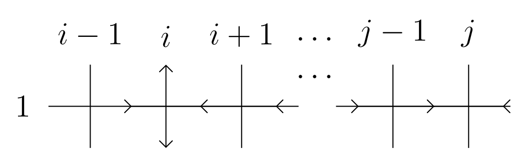
The shift invariant lemma justifies our convention to write $x_1 = 1$ as the last entry in the integer partitions.

Lemma 2.2. *Every ice state in* $\mathcal{A}^{(x_1,x_2,\cdots,x_n)}$ $(x_1 > x_2 > \cdots > x_n)$ *corresponds to a* $x_1 \times n$ *matrix, with entries from* $0, 1$ *and* -1 *satisfying the following property:*

(1) *On each row and each column, the nonzero entries alternate in sign.*
(2) *The first nonzero entry in each row is* 1.

Proof. First notice that given an ice state from $\mathcal{A}^{[x_1,x_2,\cdots,x_n]}$, we can use Kuperberg's conversion method to get a matrix whose only nonzero entries are ± 1 from the ice state.

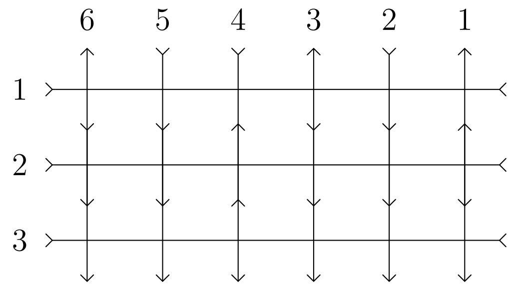
Suppose that on some row k, the first nonzero entry after $(k, i) = 1$ is $(k, j) = 1$. Then part of the ice state should look like



This means we must have a NS which correspond to a -1 in the matrix, between vertex (k, i) and vertex (k, j), but it contradicts the fact that (k, j) is the first nonzero entry after (k, i). So the first nonzero entry after 1 must be -1 and likewise the first nonzero entry after -1 must be -1. Thus the nonzero entries alternate in sign in each row.

The boundary condition says that along each row in the lattice, the leftmost arrow is always $\rightarrow$. When the first $\leftarrow$ appears on this row, we must have a EW vertex, which is a 1. Therefore the first nonzero entry on each row must be 1. □

This construction is bijective, and for example (from [**3**]), the ice state



is in bijection with the matrix

$$\begin{pmatrix} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

LEMMA 2.3. *Every ice state in $\mathcal{A}^\lambda$ corresponds to a strict Gelfand-Tsetlin pattern with fixed top row λ.*

As an example (from [**3**]), given the state above, the corresponding Gelfand-Tsetlin pattern is

$$\begin{array}{ccccc} 6 & & 3 & & 1 \\ & 4 & & 1 & \\ & & 4 & & \end{array}.$$

It is not hard to see that this construction is bijective, and this generalizes the bijection between complete Gelfand-Tsetline patterns and square ice states.

Let $x_1 = 6, x_2 = 3, x_3 = 1, y_1 = 4$ and $y_2 = 1$, then $x_1 \geq y_1 \geq x_2 \geq y_2 \geq x_3$. We express this condition by saying these two partitions *interleave*. From the definition, two consecutive rows in a Gelfand-Tsetlin pattern interleave.

CHAPTER 5

Rank 2 & 3 type $\mathcal{A}^\lambda$ ice

1. Rank 2

Lemma 1.1. *Let* $x_2 = 1$*. Then* $|\mathfrak{A}^{(x_1,x_2)}| = x_1$ *for* $x_1 \geq 2$.

Proof. We present two proofs, one using the ice models and one with the GT pattern, and because of the bijection the two methods yield the same result.

First we use ice models. Figure 1 shows the boundary condition determined by $\lambda = (x_1, 1)$. Notice that every vertex in this graph as at least one arrow filed in, Only the four corner

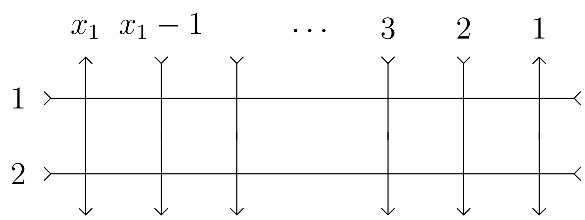
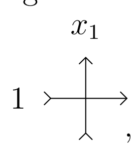


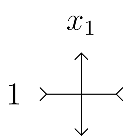
Figure 1. Boundary condition of $\mathfrak{A}^{[x_1,1]}$.

vertices have two arrows filed in. In order to have a valid six-vertex state, the out-degree and in-degree should be 2 for every vertex. For any vertex, once 3 out of 4 edges are filled in with arrows, then direction of the fourth edge is uniquely determined. So we should divide into cases according to the configuration of the top-left vertex $(1, x_1)$.

Suppose $(1, x_1)$ is SE 

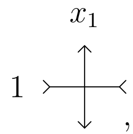
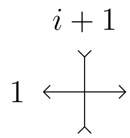
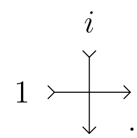
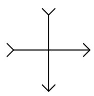
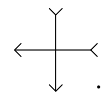
, then then all the other undetermined horizontal arrows on row 1 must be $\rightarrow$, and the ice state correspond to the matrix

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix}.$$

Suppose $(1, x_1)$ is EW 

and all other horizontal arrows on row 1 is $\leftarrow$, then the ice state is uniquely determined and correspond to the matrix

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix}.$$

Suppose $(1, x_1)$ is EW , and the arrow between nodes $(1, i+1)$ and $(1, i)$ is $\rightarrow$ for some $1 < i < k-1$, then the vertex $(1, i+1)$ is NS:  and $(1, i)$ is NW: . In order to satisfy the requirements of an ice model, all the vertices to the right of $(1, i)$ must also be  and all the vertices to the left of $(1, i+1)$ must be . Then all the arrows in this graph can be uniquely determined, so shown in figure 2. We can also write this ice states as a $2 \times k$ matrix:

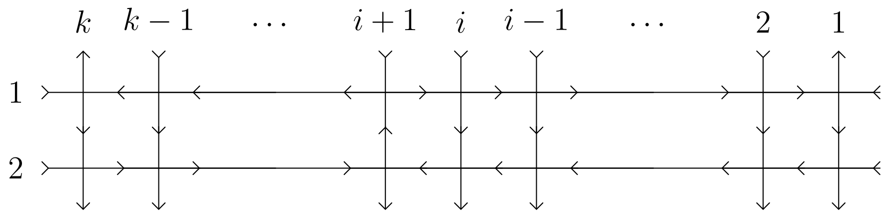


FIGURE 2. Ice model of $\mathfrak{A}^{(x_1,1)}$ with $(1, i+1) = -1$.

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

As shown above, each choice of i uniquely determines the configuration of the ice model. Note that there are $(x_1 - 2)$ choices for i, so $|\mathfrak{A}^{[x_1,1]}| = x_1 - 2$ for $x_1 \geq 2$. Therefore there are $1 + 1 + (x_1 - 2) = x_1$ ice states in total.

To prove through the GT pattern is easier. We need to prove that the number of GT patterns with top row $\lambda = (x_1, 1)$ is x_1. Notice that the GT pattern has $|\lambda| = 2$ rows, and the only entry in the second row can be $1, 2, \cdots, x_1$. So there are x_1 such GT patterns. $\square$

2. Rank 3

LEMMA 2.1. *Let $x_1 > x_2 > 1$, then*

$$|\mathcal{A}^{x_1,x_2,1}| = \frac{1}{2}\left[x_2 x_1^2 - (x_2^2 - 2x_2)x_1 + (-x_2^2 + x_2 - 2)\right].$$

PROOF. We can use the strict Gelfand-Tsetlin pattern. We want to find $y_1, y_2, z_1 \in \mathbb{N}$ such that

$$\begin{matrix} x_1 & & x_2 & & x_3 \\ & y_1 & & y_2 & \\ & & z_1 & & \end{matrix}$$

is a strict GT pattern,i.e. $x_1 \geq y_1 \leq j \geq y_2 \geq 1, y_1 \geq z_1 \geq y_2$ and $y_1 > y_2$. There are $x_1 - x_2 + 1$ possible values for y_1 and x_2 possible values for y_2, and once y_1, y_2 are chosen, there are $y_1 - y_2 + 1$ possible values for z_1. So there are

$$\sum_{y_1=x_1}^{x_2} \sum_{y_2=1}^{x_1} (y_2 - y_1 + 1)$$

possible GT patterns. But we have counted the case where $y_1 = k = y_2$ twice, so we need to substract 1 from Equation 2, which gives use the strict GT pattern enumeration

$$\begin{aligned}
|\mathcal{A}^{[x_1,x_2,1]}| &= \sum_{y_1=x_2}^{x_1} \sum_{y_2=1}^{x_2} (y_1 - y_2 + 1) - 1 \\
&= \sum_{y_1=x_2}^{x_1} \left[(y_1+1)x_2 - \frac{x_2(x_2+1)}{2} \right] - 1 \\
&= \frac{x_2}{2} \sum_{y_1=x_2}^{x_1} [2y_1 - x_2 + 1] - 1 \\
&= \frac{x_2}{2} [(x_1+x_2)(x_1-x_2+1) + (-x_2+1)(x_1-x_2+1)] - 1 \\
&= \frac{1}{2} x_2 (x_1 - x_2 + 1)(x_1 + 1) - 1
\end{aligned}$$

□

By fixing the smallest entry in the integer partition, we effectively have a enumeration for all rank 2 and 3 ice models due to the shift invariant lemma (Lemma 2.1).

COROLLARY 2.2. *For $x_1 > x_2 > x_3$ nonzero integers, we have*

$$|\mathcal{A}^{(x_1,x_2)}| = x_1 - x_2 + 1$$

and

$$|\mathcal{A}^{(x_1,x_2,x_3)}| = \frac{1}{2} [(x_2 - x_3 + 1)(x_1 - x_2 + 1)(x_1 - x_3 + 2)] - 1.$$

CHAPTER 6

Enumeration of $\mathcal{A}^\lambda$

Through the 1-to-1 correspondence, counting the states of type $\mathcal{A}^\lambda$ is the same as counting the number of integral strict Gelfand-Tsetlin pattern with the top row λ. As we can see in the rank 2 and rank 3 case, investigating the strict GT patterns is easier for enumeration purposes.

The goal of the thesis is to supply a formula $F_{(n)} : \mathbb{Z}^n \to \mathbb{Z}$ such that $F_{(n)}(\lambda) = |\mathcal{A}^\lambda|$ where $|\lambda| = n$ for any $n \in \mathbb{N}$.

1. Recursive relationship

We give a recursive relationship that connects $F_{(k)}$ and $F_{(k+1)}$.

THEOREM 1.1. *The number of $\mathcal{A}^\lambda$ ice model, which is the same as the number of strict Gelfand-Tsetline patterns with top row λ, where $\lambda = (x_1, x_2, \cdots, x_n)$ strictly decreasing, is given by*

$$\begin{aligned} F_{(n)}(x_1, x_2, \cdots, x_n) &= \sum_{\substack{x_1 \geq y_1 \geq x_2 \geq y_2 \geq x_3 \geq \cdots y_{n-1} \geq x_n \\ y_1 > y_2 > \cdots > y_{n-1}}} F_{(n-1)}(y_1, y_2, \cdots, y_{n-1}) \\ &= \sum_{y_1 = x_2}^{x_1} \sum_{y_2 = x_3}^{x_2} \cdots \sum_{y_{n-1} = x_n}^{x_{n-1}} \left[\left[\prod_{i=1}^{n-2} [1 - \delta(y_i, y_{i+1})] \right] F_{(n-1)}(y_1, y_2, \cdots, y_{n-1}) \right]. \end{aligned}$$

Note that δ is the Kronecker-delta and the base case is $F_{(1)}(x) = 1$.

PROOF. Clearly when $n = 1$, there can be only one ice model, so $F_{(1)}(x) = 1$ for all $n \in \mathbb{N}$. When we are given the first row $x_1, x_2, \cdots, x_n$, the second row $y_1, y_2, y_3 \cdots y_{n-1}$ in a strict Gelfand Tsetlin pattern

$$\begin{array}{ccccccccccccc} x_1 & & x_2 & & x_3 & & \cdot & & \cdot & & \cdot & & x_n \\ & y_1 & & y_2 & & \cdot & & \cdot & & \cdot & & y_n & \end{array}$$

must satisfy

- (Condition 1) $x_1 \geq y_1 \geq x_2 \geq y_2 \geq x_3 \geq \cdots y_{n-1} \geq x_n$ and
- (Condition 2) $y_1 > y_2 > \cdots > y_{n-1}$.

Each possible occurrence of $y_1, y_2, \cdots, y_{n-1}$ is a top row of a rank $(n-1)$ strict Gelfand-Tsetline pattern, and there are $F_{(n-1)}(y_1, y_2, \cdots, y_{n-1})$ of them. If we add up all results given by possible second row, then we get

$$F_{(n)}(x_1, x_2, \cdots, x_n) = \sum_{\substack{x_1 \geq y_1 \geq x_2 \geq y_2 \geq x_3 \geq \cdots y_{n-1} \geq x_n \\ y_1 > y_2 > \cdots > y_{n-1}}} F_{(n-1)}(y_1, y_2, \cdots, y_{n-1}). \tag{12}$$

To prove the second part, note that we can rewrite the second condition as $y_i \neq y_j$ for all $1 \leq i, j \leq n-1$. In fact, since λ is strictly decreasing, then we only need $y_i \neq y_{i+1}$ for all $1 \leq i \leq n-2$.

We can add $F_{(n-1)}(y_1, y_2, \cdots, y_{n-1})$ from all possible states that satisfy the first condition, but add 0 when Condition 2 is not met. The Kronecker-Delta function is useful for this purpose because Condition 2 fails if and only if $y_i = y_{i+1}$ for some i. Recall that

Definition 11. The Kronecker-Delta is a function of two variables

$$\delta : \mathbb{R}^2 \to \{0, 1\}$$

such that

$$\delta(x, y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise.} \end{cases}$$

If Condition 2 fails, then

$$0 = \prod_{i=1}^{n-2} [1 - \delta(y_i, y_{i+1})] = \prod_{i=1}^{n-2} [1 - \delta(y_i, y_{i+1})] \, F_{(n-1)}(y_1, y_2, \cdots, y_{n-1}). \tag{13}$$

Collect all the collisions on row 1, we get

$$\sum_{\substack{x_1 \geq y_1 \geq x_2 \geq y_2 \geq x_3 \geq \cdots y_{n-1} \geq x_n \\ y_i = y_{i+1} \text{ for some } i}} \left[\prod_{i=1}^{n-2} [1 - \delta(y_i, y_{i+1})] \, F_{(n-1)}(y_1, y_2, \cdots, y_{n-1}) \right] = 0 \tag{14}$$

If the second condition $y_i \neq y_{i+1}$ is satisfied, then

$$\prod_{i=1}^{n-2} [1 - \delta(y_i, y_{i+1})] = 1. \tag{15}$$

So

$$\left[\prod_{i=1}^{n-2} [1 - \delta(y_i, y_{i+1})] \right] F_{(n+1)}(y_1, y_2, \cdots, y_{n-1}) = F_{(n+1)}(y_1, y_2, \cdots, y_{n-1}). \tag{16}$$

If we add Equation 14 to Equation 12, then we have

$$F_{(n)}(x_1, x_2, \cdots, x_n) = \sum_{\substack{x_1 \geq y_1 \geq x_2 \geq y_2 \geq x_3 \geq \cdots y_{n-1} \geq x_n \\ y_1 > y_2 > \cdots > y_{n-1}}} F_{(n-1)}(y_1, y_2, \cdots, y_{n-1}) + 0 \tag{17}$$

$$= \sum_{\substack{x_1 \geq y_1 \geq x_2 \geq y_2 \geq x_3 \geq \cdots y_{n-1} \geq x_n \\ y_1 > y_2 > \cdots > y_{n-1}}} F_{(n-1)}(y_1, y_2, \cdots, y_{n-1}) \tag{18}$$

$$+ \sum_{\substack{x_1 \geq y_1 \geq x_2 \geq y_2 \geq x_3 \geq \cdots y_{n-1} \geq x_n \\ y_i = y_{i+1} \text{ for some } i}} \left[\left[\prod_{i=1}^{n-2} [1 - \delta(y_i, y_{i+1})] \right] F_{(n+1)}(y_1, y_2, \cdots, y_{n-1}) \right] \tag{19}$$

$$= \sum_{\substack{x_1 \geq y_1 \geq x_2 \geq y_2 \geq x_3 \geq \cdots y_{n-1} \geq x_n \\ y_1 > y_2 > \cdots > y_{n-1}}} \left[\left[\prod_{i=1}^{n-2} [1 - \delta(y_i, y_{i+1})] \right] F_{(n+1)}(y_1, y_2, \cdots, y_{n-1}) \right] \tag{20}$$

$$+ \sum_{\substack{x_1 \geq y_1 \geq x_2 \geq y_2 \geq x_3 \geq \cdots y_{n-1} \geq x_n \\ y_i = y_{i+1} \text{ for some } i}} \left[\left[\prod_{i=1}^{n-2} [1 - \delta(y_i, y_{i+1})] \right] F_{(n+1)}(y_1, y_2, \cdots, y_{n-1}) \right] \tag{21}$$

$$= \sum_{y_1 = x_2}^{x_1} \sum_{y_2 = x_3}^{x_2} \cdots \sum_{y_{n-1} = x_n}^{x_{n-1}} \left[\left[\prod_{i=1}^{n-2} [1 - \delta(y_i, y_{i+1})] \right] F_{(n+1)}(y_1, y_2, \cdots, y_{n-1}) \right]. \tag{22}$$

□

In general, the closed form formula is difficult to find, but we have found the closed form expression for small enough n, for example:

$$\begin{aligned} F_{(2)}(x_1, x_2) &= x_1 - x_2 + 1; \\ F_{(3)}(x_1, x_2, x_3) &= \frac{1}{2} [(x_2 - x_3 + 1)(x_1 - x_2 + 1)(x_1 - x_3 + 2)] - 1 \\ &= \frac{1}{2} [(x_2 - x_3 + 1) x_1^2 + \left(-x_2^2 + 2x_2 - 4x_3 + x_3^2 + 3\right) x_1 \\ &+ \left(x_2^2 x_3 - x_2 x_3^2 - 2x_2^2 + x_3^2 + 2x_2 x_3 - 3x_3\right)] \\ &= x_1^2 x_2 - x_1 x_2^2 - x_1^2 x_3 + x_2^2 x_3 + x_1 x_3^2 - x_2 x_3^2 + x_1^2 + 2x_1 x_2 - 2x_2^2 - 4x_1 x_3 \\ &+ 2x_2 x_3 + 3x_1 - 3x_3. \end{aligned}$$

I have proved $F_{(2)}$ and $F_{(3)}$ using elementary combinatorics in the previous section. To demonstrate the power of Theorem 5.1,

$$F_{(4)}(x_1, x_2, x_3, x_4) = \sum_{y_3 = x_4}^{x_3} \sum_{y_2 = x_3}^{x_2} \sum_{y_1 = x_2}^{x_1} F_{(3)}(y_1, y_2, y_3) - \sum_{y_1 = x_2}^{x_1} F_{(3)}(y_1, x_3, x_3) - \sum_{y_3 = x_4}^{x_3} F_{(3)}(x_2, x_2, y_3). \tag{23}$$

Sage tells us that

$$\begin{aligned} \sum_{y_3 = x_4}^{x_3} \sum_{y_2 = x_3}^{x_2} \sum_{y_1 = x_2}^{x_1} F_{(3)}(y_1, y_2, y_3) = \frac{1}{12} &(x_1 - x_2 + 1)(x_2 - x_3 + 1)(x_3 - x_4 + 1)(x_1^2 x_2 - x_1 x_2 x_3 \\ &- x_1^2 x_4 - x_1 x_2 x_4 + x_1 x_3 x_4 + x_2 x_3 x_4 + x_1 x_4^2 - x_3 x_4^2 \\ &+ 2x_1^2 + 5x_1 x_2 - 2x_1 x_3 - 3x_2 x_3 - 7x_1 x_4 - 2x_2 x_4 + 5x_3 x_4 \\ &+ 2x_4^2 + 10x_1 + 6x_2 - 6x_3 - 10x_4), \end{aligned} \tag{24}$$

$$\sum_{y_1=x_2}^{x_1} F_{(3)}(y_1, x_3, x_3) = \frac{1}{6}(x_3 - x_4 + 1)(3x_2^2 - 3x_2x_3 + x_3^2 + 3x_2x_4 + x_3x_4 + x_4^2 + 9x_2 - 4x_3 - 5x_4), \tag{25}$$

and

$$\sum_{y_3=x_4}^{x_3} F_{(3)}(x_2, x_2, y_3) = \frac{1}{6}(x_1 - x_2 + 1)(x_1^2 + x_1x_2 + x_2^2 - 3x_1x_3 - 3x_2x_3 + 3x_3^2 + 5x_1 + 4x - 2 - 9x_3). \tag{26}$$

Plugging Equation 24-26 into Equation 23, we get

$$\begin{aligned} F_{(4)}(x_1, x_2, x_3, x_4) = \frac{1}{12}(&x_1^3 + x_2^2x_3 - x_1^3x_2x3^2 - x_1^3x_2^3x_3^2 \\ &+ x_1^2x_2x_3^3 - x_1x_2^2x_3^3 - x_1^3x_2^2x_4 + x_1^2x_2^3x_4 + x_1^3x_3^2x_4 \\ &- x_2^3x_3^2x_4 - x_1^2x_3^3x_4 + x_2^2x_3^3x_4 + x_1^3x_2x_4^2 \\ &+ x_1x_2^3x_4^2 - x_1^3x_3x_4^2 + x_2^3x_3x_4^2 + x_1x_3^3x_4^2 - x_2x_3^3x_4^2 \\ &- x_1^2x_2x_4^3 + x_1x_2^2x_4^3 + x_1^2x_3x_4^3 - x_2^2x_3x_4^3 \\ &+ x_1x_3^2x_4^3 + x_2x_3^2x_4^3 + x_1^3x_2^2 - x_1^2x_2^3 + 2x_1^3x_2x_3 \\ &+ 3x_1^2x_2^2x_3 - 4x_1x_2^3x_3 - 2x_1^3x_3^2 - 6x_1^2x_2x_3^2 \\ &+ 3x_1x_2^2x_3^2 + 3x_2^3x_3^2 + 2x_1^2x_3^3 + 2x_1^2x_2x_3^3 - 3x_2^2x_3^3 \\ &- 4x_1^3x_2x_4 - 3x_1^2x_2^2x_4 + 6x_1x_2^3x_4 + 2x_1^3x_3x_4 \\ &- 2x_2^3x_3x_4 + 6x_1^2x_3^2x_4 - 3x_2^2x_3^2x_4 - 6x_1x_3^3x_4 \\ &+ 4x_2x_3^3x_4 + x_1^3x_4^2 + 9x_1^2x_2x_4^2 - 6x_1x_2^2x_4^2 - 2x_2^3x_4^2 \\ &- 9x_1^2x_3^3x_4^2 + 6x_2^2x_3x_4^2 + 3x_1x_3^2x_4^2 - 3x_2x_3^2x_4^2 \\ &+ x_3^3x_4^2 - x_1^2x_4^3 - 2x_1^3x_2 + 3x_1^2x_2^2 - 5x_1x_2^3 \\ &+ 2x_2x_3x_4^3 - x_3^2x_4^3 + 3x_1^3x_2 + 3x_1^2x_2^2 - 5x_1x_2^3 \\ &+ 9x_1^2x_2x_3 - 3x_2^3x_3 - 12x_1^2x_3^2 - 9x_1x_2x_3^2 - 5x_1x_2^3 \\ &+ 8x_1x_3^3 - 3x_2x_3^3 - 3x_1^3x_4 - 24x_1^2x_2x_4 + 9x_1x_2^2x_4 \\ &+ 8x_2^3x_4 + 15x_1^2x_3x_4 - 9x_2^2x_3x_4 + 9 + 9x_1x_3^2x_4 - 5x_3^3x_4 \\ &+ 9x_1^2x_4^2 + 15x_1x_2x_4^2 - 12x_2^2x_4^2 - 24x_1x_3x_4^2 \\ &+ 9x_2x_3x_4^2 + 3x_3^2x_4^2 - 3x_1x_4^3 + 3x_3x_4^3 + 16x_1^2x_2 \\ &- 4x_1x_2^2 - 4x_2^3 + 4x_1^2x_3 - 16x_1x_3^2 + 4x_3^3 - 20x_1^2x_4 \\ &- 24x_1x_2x_4 + 16x_2^2x_4 + 24x - 1x_3x_4 + 4x_3^2x_4 + 20x_1x_4^2 \\ &- 4x_2x_4^2 - 16x_3x_4^2 + 11x_1x_2 + 16x_1x_3 - 27x_2x_3 - 27x_1x_4 \\ &+ 16x_2x_4 + 11x_3x_4 - 20x_2 + 20x_3). \end{aligned} \tag{27}$$

2. Enumeration of Gelfand-Tsetlin Pattern

When we construct the above recursive relationship, we have to multiply by some kronecker-delta, to make sure that we never count the case where two adjacent entries in the same row are the same. As we have established in previous sections, entries in a Gelfand-Tsetlin pattern correspond to occurrence of up-arrows in the ice model, and therefore only strict Gelfand-Tsetlin patterns correspond to ice models. If we remove this strictly-increasing condition, we no longer have ice models, but there are still Gelfand-Tsetlin patterns and their matrix correspondence.

Firstly we are interested in an enumeration. Let λ be an integer partition. Define

$$G_{(n)} : \mathbb{Z}^n \to \mathbb{Z}$$

given by $G(\lambda) = \#$ of Gelfand Tsetlin patterns whose top row is the integer partition λ. We have an recursive formula:

$$G_{(n)}(x_1, x_2, \cdots, x_n) = \sum_{y_1=x_2}^{x_1} \sum_{y_2=x_3}^{x_2} \cdots \sum_{y_{n-1}=x_n}^{x_{n-1}} \left[G_{(n-1)}(y_1, y_2, \cdots, y_{n-1})\right] \tag{28}$$

and base case is $G_{(1)}(x) = 1$. There is closed form formula and Kuperberg proved it using induction in [**4**].

THEOREM 2.1 (Enumeration of Gelfand-Tsetlin Pattern). *Let $x_1 > x_2 > \cdots > x_n$ be positive integers. Then*

$$G(x_1, x_2, \cdots, x_n) = \prod_{1 \le i < j \le n} \frac{x_i - x_j + j - i}{j - i}. \tag{29}$$

We can also use Theorem 1.1 to derive the formula for $F_{(4)}(x_1, x_2, x_3, x_4)$.

PROOF. It is not hard to see that

$$F_{(4)}(x_1, x_2, x_3, x_4) = G_{(4)}(x_1, x_2, x_3, x_4) - C(x_1, x_2, x_3, x_4) \tag{30}$$

where $C(x_1, x_2, x_3, x_4)$ is number of states with some collisions.

If we write out the Gelfand-Tsetline pattern

$$\begin{array}{ccccccc} x_{1,1} & & x_{1,2} & & x_{1,3} & & x_{1,4} \\ & x_{2,2} & & x_{2,3} & & x_{2,4} & \\ & & x_{3,3} & & x_{3,4} & & \\ & & & x_{4,4} & & & \end{array},$$

we can see that collisions can only happen at $(1, 2), (1, 3)$ and $(2, 3)$, and we cannot have collisions at $(1, 2), (1, 3)$ at the same time. We can see that there are 3 different cases:

(1) There is collision at $(1, 2)$,i.e. $x_{1,2} = x_{2,2} = x_{2,3} = x_{3,3}$ and there might be collision at $(2, 3)$ as well.
(2) There is collision at $(1, 3)$,i.e. $x_{1,3} = x_{2,3} = x_{2,4} = x_{3,4}$ and there might be collision at $(2, 3)$ as well.
(3) There is no collision on row 2, but a collision at $(2, 3)$, i.e. $x_{2,3} = x_{3,3} = x_{3,4} = x_{4,4}$.

Let c_i denote the number of Gelfand-Tsetlin pattern with collision in case i, then we have $C(x_1, x_2, x_3, x_4) = c_1 + c_2 + c_3$. For convinience, we write the rank 4 Gelfand-Tsetlin pattern as

$$\begin{array}{ccccccc} x_1 & & x_2 & & x_3 & & x_4 \\ & y_1 & & y_2 & & y_3 & \\ & & z_1 & & z_2 & & \\ & & & w_1 & & & \end{array}.$$

Then the 3 cases can be illustrated as:

(1) Case 1:

$$\begin{array}{ccccccc} x_1 & & x_2 & & x_3 & & x_4 \\ & x_2 & & x_2 & & y_3 & \\ & & x_2 & & z_2 & & \\ & & & w_1 & & & \end{array} \quad \text{or} \quad \begin{array}{ccccccc} x_1 & & x_2 & & x_3 & & x_4 \\ & x_2 & & x_2 & & y_3 & \\ & & x_2 & & x_2 & & \\ & & & x_2 & & & \end{array}.$$

Case 1 contains all Gelfand-Tsetlin patterns with top row x_2, x_2, y_3 where $x_4 \leq y_3 \leq x_3$, so

$$\begin{aligned} c_1 &= \sum_{y=x_4}^{x_3} G_{(3)}(x_2, x_2, y_3) \\ &= \sum_{y=x_4}^{x_3} \frac{(x_2 - y_3 + 1)(x_2 - y_3 + 2)(x_2 - x_2 + 1)}{2} \\ &= \sum_{y=x_4}^{x_3} \frac{1}{2}(x_2 - y_3 + 1)(x_2 - y_3 + 2) \\ &= \frac{1}{6}(x_3 - x_4 + 1)(3x_2^2 - 3x_2x_3 + x_3^2 - 3x_2x_4 + x_3x_4 + x_4^2 + 9x_2 - 4x_3 - 5x_4 + 6); \end{aligned} \tag{31}$$

(2) Case 2:

$$\begin{array}{ccccccc} x_1 & & x_2 & & x_3 & & x_4 \\ & y_1 & & x_3 & & x_3 & \\ & & z_1 & & x_3 & & \\ & & & w_1 & & & \end{array} \quad \textbf{or} \quad \begin{array}{ccccccc} x_1 & & x_2 & & x_3 & & x_4 \\ & y_1 & & x_3 & & x_3 & \\ & & x_3 & & x_3 & & \\ & & & x_3 & & & \end{array}.$$

Case 1 contains all Gelfand-Tsetlin patterns with top row y_1, x_3, x_3 where $x_1 \leq y_1 \leq$

y_2, so

$$\begin{aligned} c_2 &= \sum_{y_1=x_2}^{x_1} G_{(3)}(y_1, x_3, x_3) \\ &= \sum_{y_1=x_2}^{x_1} \frac{(y_1-x_3+1)(y_1-x_3+2)(x_3-x_3+1)}{2} \\ &= \frac{1}{2}\sum_{y_1=x_2}^{x_1}(y_1-x_3+1)(y_1-x_3+2) \\ &= \frac{1}{6}(x_1-x_2+1)(x_1^2+x_1x_2+x_2^2-3x_1x_3-3x_2x_3+3x_3^2+5x_1+4x_2-9x_3+6). \end{aligned} \tag{32}$$

(3) Case 3: $\begin{array}{ccccccc} x_1 & & x_2 & & x_3 & & x_4 \\ & y_1 & & y_2 & & y_3 & \\ & & y_2 & & y_2 & & \\ & & & y_2 & & & \end{array}$.

We pick y_2 first. If $y_1 < y_2 < y_3$, then there are (x_1-x_2+1) choices for y_1, (x_2-x_3-1) choices for y_2 and (x_3-x_4+1) choices are y_3. If $y_2 = x_2$, then there are (x_1-x_2) choices for y_1 and (x_3-x_4+1) choices for y_3. The $y_2 = x_3$ case is similar. So

$$c_3 = (x_2-x_3-1)(x_1-x_2+1)(x_3-x_4+1)+(x_1-x_2)(x_3-x_4+1)+(x_1-x_2+1)(x_3-x_4) \tag{33}$$

Combining Equations 29 through 33, we can get the same result as Equation 27. □

CHAPTER 7

Characters of Classical Groups

1. Okada

In [**10**], Okada showed that "the partition functions corresponding to the round 1-,2- and 3-enumerations are expressed in terms of irreducible characters of classical groups up to simple factors" and he obtained the enumeration formulae for some symmetry classes of ASM.

THEOREM 1.1 (Okada,[**10**]). *The number of $n \times n$ ASMs is given by*

$$A_n = 3^{-n(n-1)/2} \dim \mathbf{GL}_{2n}(\delta(n-1, n-1)). \tag{34}$$

where $\dim \mathbf{GL}(\delta)$ *denotes the dimension of the irreducible representation of* $\mathbf{GL}_N$ *with the "heightest weight"* λ*, and*

$$\delta(n-1, n-1) = (n-1, n-1, n-2, n-2, \cdots, 2, 2, 1, 1).$$

Okada's Theorem 1.1 implies the connection between representation theory and this enumeration problem. As we have mentioned in Chapter 3, the enumeration of $n \times n$ ASMs relates to monotone triangles, which show up as Gelfand-Tsetlin pattern in representation theory.

2. Rank 3 Type $\mathcal{A}$ Ice Model

We want to find a formula similar to Equation 34 that relates $|A^{(x_1,x_2,x_3)}| = F_{(3)}(x_1, x_2, x_3)$ to irreducible characters of classical lie algebra.

Without loss of generality, we can assume $x_3 = 1$.

CONJECTURE 1. *Let* $x_1 > x_2 > 1$*, then*

$$\frac{\dim \mathbf{GL}_3(x_1, x_2, 1, 1)}{F_{(3)}(x_1, x_2, 1) + 1} = \frac{(x_1^3 + 11x_1^2 + 38x_1 + 40)(x_2^4 + 10x_2^3 + 35x_2^2 + 50x_2 + 24)}{240(x_2 + 2)}.$$

We obtain the above formula from simulation in the mathematical software Sage, and have checked the result is true for x_1 up to 18.

3. Forward

This thesis present my one-year work trying to enumerate type $\mathcal{A}^\lambda$ ice models and strict Gelfand-Tsetlin patterns. The current recursive formula we have is inadequate, and we still seek a formula like Equation 29.

Many combinatorial ideas presented in this thesis play an important role in the study of representation theory and lie algebras. We believe that future investigate on enumeration of strict Gelfand-Tsetlin patterns should be in the direction of relating these combinatorial problems to representation theory.

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