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Dynamic scaling in stick-slip friction

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We introduce a generalized homogeneous function to describe the joint probability density for magnitude and duration of events in self-organized critical systems (SOC). It follows that the cumulative distributions of magnitude and of duration are power-laws with exponents α and τ respectively. A power-law relates duration and magnitude (exponent γ) on the average. The exponents satisfy the dynamic scaling relation $\alpha = \gamma\tau$. The exponents classify SOC systems into universality classes that do not depend on microscopic details provided that both $\alpha \leq 1$ and $\tau \leq 1$. We also present new experimental results on the stick-slip motion of a sandpaper slowly pulled across a carpet that are consistent with our criteria for SOC systems. Our experiments, as well as experiments by others, satisfy our dynamic scaling relation. We discuss the relevance of our results to earthquake statistics.

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INTRODUCTION

The force required to pull a block along a surface is proportional to its weight and independent of contact area. The sliding motion of the block depends on the elastic properties of the string pulling the block, the block itself, and the surface, and takes many forms [1, 2, 3]. For a stiff system, periodic slips with a velocity dependent amplitude have been observed [4, 5]. Intermittent stick-slip friction has been observed in experiments where a highly compliant string was used to pull a piece of sandpaper across a soft carpet [6]. We extend these measurements to resolve the slip duration and find that our system is an example of a self-organized critical (SOC) system [7].

We propose a testable criterion for scale-free SOC systems: The joint probability density of slip magnitude and duration is a generalized homogeneous function, the exponents of the cumulative magnitude and duration distributions must be less than unity, so that moments of the distributions do not exist, and a dynamic scaling relation between the exponents must be satisfied. It follows that periodic stick-slip motion [4, 5] cannot be SOC. The experimental results presented here satisfy the definition of SOC accurately.

The scaling relation is also satisfied by the magnitude and duration exponents of rain showers [8], and by exponents for slips in a granular medium [9]. In our view, earthquakes are not SOC since the currently accepted exponents are larger than one.

EXPERIMENTS

A weighted sandpaper was pulled by an elastic nylon line slowly (9 μm/s) across a carpet [10]. The sandpaper moved by a stick-slip process. We measured the force required to hold the carpet in place (see Fig. 1) at intervals of $\delta t = (14\text{Hz})^{-1}$, that is, every 0.6 μm on the average. For each slip, we observed an abrupt decrease in force, ΔF, and for the first time we have been able to measure the duration, t, of slips.

The force per unit area (stress), $\boldsymbol{f}(\boldsymbol{r})$, between the sandpaper and the carpet is a function of position, $\boldsymbol{r}$, within the surface area, A, of the sandpaper. The integral of the vertical component of $\boldsymbol{f}(\boldsymbol{r})$ is a constant equal to the weight of the sandpaper slider (mass m): $mg = \int_A \boldsymbol{f}(\boldsymbol{r}) \cdot \boldsymbol{n}\, \mathrm{d}^2\boldsymbol{r}$, where g is the acceleration of gravity, and $\boldsymbol{n}$ is the surface normal. In a slip, the contact between the sandpaper and the carpet is lost, or reduced, over an area $S \leq A$, that is not necessarily connected. The force changes by $\delta\boldsymbol{f}(\boldsymbol{r}) = \mu\, \boldsymbol{u}(\boldsymbol{r})/h$, where $\boldsymbol{u}(\boldsymbol{r})$ is the local slip vector, in the plane of the carpet, h is the carpet thickness, μ is the shear modulus of the carpet, and $\boldsymbol{u}(\boldsymbol{r})/h$ is the slip-associated change in shear strain at $\boldsymbol{r}$. The slip leads to an elastic increase in shear strain, $\boldsymbol{u}_e(\boldsymbol{r})/h$ outside S, and increased elastic stress takes more of the external load. Integrating $\delta\boldsymbol{f}(\boldsymbol{r})$ over the area of the sandpaper, A, we find that

$$\begin{aligned} m = h\Delta F &= h \int_A \delta\boldsymbol{f}(\boldsymbol{r}) \cdot \mathrm{d}^2\boldsymbol{r} \\ &= \mu \int_S \boldsymbol{u}(\boldsymbol{r}) \cdot \mathrm{d}^2\boldsymbol{r} + \mu \int_{A-S} \boldsymbol{u}_e(\boldsymbol{r}) \cdot \mathrm{d}^2\boldsymbol{r} \\ &= \mu' S \langle u \rangle_S \ . \end{aligned}$$

Experimentally we measure the first integral. The next integrals represent a model where slip occurs over an area S, and an increased elastic strain elsewhere in A. The integral over $A - S$ is proportional to the integral over S, and $\langle u \rangle_S$ is the mean slip averaged over the slip area S. The last expression is the standard form of the (scalar) seismic moment M_0 with an effective shear modulus μ' [11, 12]. We conclude that by measuring ΔF we obtain the seismic moment, m, of the slip directly.

A slip starts at t_1 if the force difference between two consecutive measurements $\delta F(t_1) = F(t_1 + t) - F(t_1)$, changes sign from positive to negative at t_1, and ends at $t_2 = t_1 + t$ when $\delta F(t_2)$ becomes positive. A slip is therefore characterized by the decrease in force $\Delta F =$

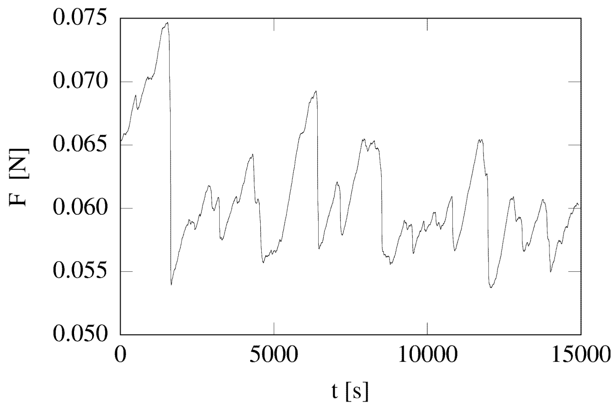

FIG. 1: Force as a function of time from an experiment. The initial rise and drop in force at the beginning of the experiment is characteristic of the interval of time in which the system "forgets" its initial state. Later slips of size ΔF and duration t, are studied here.

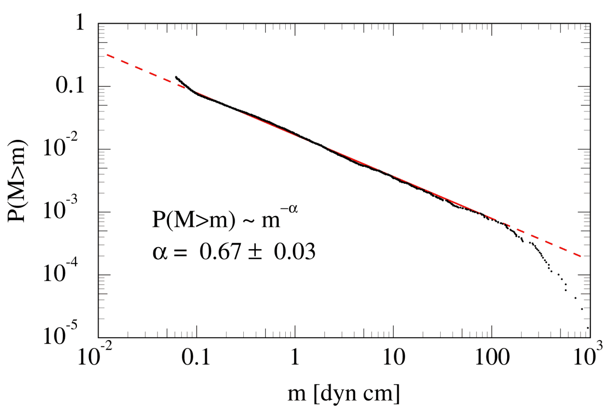

FIG. 2: The cumulative distribution, $P(M > m)$, as a function of the seismic moment, m. The observed cumulative distribution is consistent with a power-law, $P(M > m) \sim m^{-\alpha}$, with $\alpha = 0.67 \pm 0.03$. The straight line has a slope of -0.67.

$F(t_1) - F(t_2)$, that is, by the seismic moment $m = h\Delta F$ and duration $t = t_2 - t_1$. We found that this procedure gave the same scaling exponents as more complicated ways of estimating slip magnitude and duration, such as sliding average and other filtering procedures, that we have tested. Of course, each slip consists of events whereby fibers lose their grip on the sandpaper one by one; the sound from slips was easy to hear. From this point of view the minimum slip consist of one single fiber losing its grip. We have seen that there are bursts of such micro-slips in the slips identified by us. There was some very low activity even in the intervals between slips. Of course, electrical noise from the strain-gauge, and measurement system limits the resolution so that we cannot separate the smallest slips from electrical noise. For the carpet used in the experiments reported here, in no case did all fibers lose their grip on the sandpaper, so that

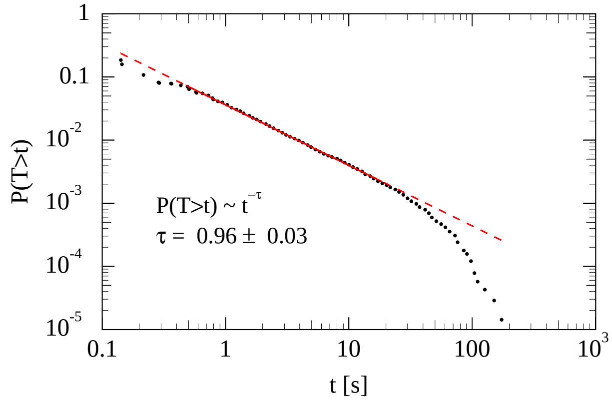

FIG. 3: The cumulative distribution, $P(T > t)$, as a function of slip duration t. The observed cumulative distribution is consistent with a power-law, $P(T > t) \sim t^{-\tau}$, with $\tau = 0.96 \pm 0.03$. The straight line has slope -0.96.

the sandpaper was never displaced more than one fiber length with respect to the carpet. In the case of carpets for which this did sometimes occur, a power-law distribution of magnitudes and durations was not observed.

Figure 2 shows that the cumulative distribution of magnitude, averaged over four independent experiments with a total of more than 50 000 slips, is consistent with the scaling form [13]

$$P(M > m) \sim m^{-\alpha} , \qquad \alpha = 0.67 \pm 0.03 . \tag{1}$$

The deviations from the power-law fit (dashed line segments were not part of the fit) at large m are due to finite-size effects, and the increase at small m is due to noise, which masquerades as small slips. Similar deviations for small and large values are seen in Figs. 3 and 4. Since power-law distributions are stable under addition [14], the effect of noise and finite sampling rate does not affect the slope of the distributions, but reduces the range over which scaling may be observed.

Figure 3 shows that the cumulative distribution of event duration time t is consistent with the scaling form

$$P(T > t) \sim t^{-\tau} , \qquad \tau = 0.96 \pm 0.03 . \tag{2}$$

We also found that slips with a large seismic moment lasted longer than small slips. Figure 4 shows the estimates of most probable values (m_0, t_0) of the joint probability density $p(m, t)$ as points, and estimates of the standard deviations (for bins that contain more than six measurements) around these points as the shaded area, that is, we have estimated t_0 to be the average duration given $m = m_0$.

The results shown in Fig. 4 are consistent with the power-law form relating the most probable pairs (m_0, t_0):

$$t_0 \sim m_0^{\gamma} , \qquad \gamma = 0.69 \pm 0.03 , \tag{3}$$

for $m > 0.15$ dyn cm.

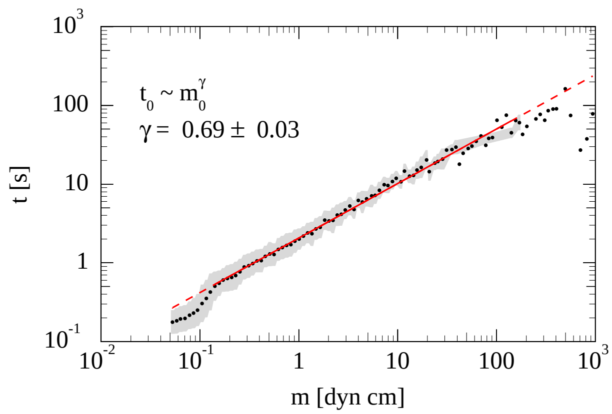

FIG. 4: The joint probability density, $p(m,t)$, of seismic moment, m, and slip duration, t, is illustrated by showing the most probable values (m_0, t_0) as points. The shaded area represents (m,t) values within one standard deviation around (m_0, t_0). The straight line is a fit of $t_0 \sim m_0^\gamma$ with a slope $\gamma = 0.69$.

THEORY

SOC, as originally defined [7], requires power-law scaling both in magnitude and duration of the events. We show that assuming the joint probability density $p(m,t)$ to be a generalized homogeneous function, it follows that the magnitude and duration distributions are power-laws, and average duration as a function of magnitude is also a power-law. The measured exponents satisfy a scaling relation that follows from the assumption that the joint probability density, $p(m,t)$, is a generalized homogeneous function, that is, for any positive value of λ:

$$p(m,t)\mathrm{d}m\,\mathrm{d}t = \lambda^{-\alpha} p\left(\frac{m}{\lambda}, \frac{t}{\lambda^\gamma}\right) \mathrm{d}\left(\frac{m}{\lambda}\right) \mathrm{d}\left(\frac{t}{\lambda^\gamma}\right) \ . \quad (4)$$

Here λ is a number that represents the change of scale provided m and t are given in dimensionless units. Such functions have been discussed extensively in the context of scaling at the critical point of second order phase transitions [15]. However, for non-equilibrium systems, of the kind discussed here, the theoretical basis for (4) has not been established and it thus remains a proposal.

The probability density, $p(m)$, for slip magnitude is obtained from eq. (4) by integrating over the slip duration

$$p(m)\mathrm{d}m = \int_{t=0}^{\infty} p(m,t)\mathrm{d}m\,\mathrm{d}t = \lambda^{-(1+\alpha)}\mathrm{d}m \int_{v=0}^{\infty} p\left(\frac{m}{\lambda}, v\right) \mathrm{d}v \ ,$$

where we have introduced the integration variable $v = t/\lambda^\gamma$. Since λ may be chosen to have any positive value we set $\lambda = m$, and find

$$p(m) = m^{-(1+\alpha)} \int_{v=0}^{\infty} p(1,v)\mathrm{d}v \sim m^{-(1+\alpha)} \ .$$

Here the integral is a constant, and it follows that the cumulative distribution, $P(M > m) = \int_m^\infty p(M)\,\mathrm{d}M \sim m^{-\alpha}$, has the scaling form given in eq. (1).

The probability density, $p(t)$, for the slip duration t is found, in a similar way, by integrating eq. (4) over m:

$$p(t) = t^{-(1+\alpha/\gamma)} \int_{v=0}^{\infty} p(v,1)\,\mathrm{d}v \sim t^{-(1+\alpha/\gamma)} \ .$$

Again the integral is a constant, and it follows that the cumulative distribution, $P(T > t) = \int_t^\infty p(T)\,\mathrm{d}T \sim t^{-\alpha/\gamma}$, has the scaling form given in eq. (2) with $\tau = \alpha/\gamma$, that is, we have the dynamic scaling relation

$$\alpha = \gamma\tau \ . \quad (5)$$

This relation also follows by assuming the scaling forms (1)–(3) and requiring consistency for the conditional expectation values [16].

Equation (3) follows from our proposition (4). Let (m_0^*, t_0^*) be a (m,t) pair where $p(m,t)$ has a local maximum corresponding to the mostprobable value for m given $t = t_0^*$ or t given $m = m_0^*$. Then, for the generalized homogeneous function (4), $m_0 = m_0^*/\lambda$ and $t_0 = t_0^*/\lambda^\gamma$ also corresponds to a local maximum in the joint probability density for any positive value of λ; it follows that

$$1/\lambda^\gamma = t_0/t_0^* = (m_0/m_0^*)^\gamma \ ,$$

which is indeed eq. (3). Since both m and t are taken to be dimensionless, we may chose both m_0^* and t_0^* to be unity for the purposes of the scaling relations. Equation (4), states that the joint probability density is invariant with respect to the affine transformation $m \to m/\lambda$, $t \to t/\lambda^\gamma$, and $p \to p/\lambda^\alpha$; it follows that any expectation and other characteristic, such as the full width at half height, will scale and have exponents that only depend on α and γ. It follows that the width of the distribution in Fig. 4 should be constant in the log-log plot.

If there is a characteristic magnitude, or duration, of slip events, then any model of the process must include the characteristic values. However, if $\alpha \leq 1$ and $\tau \leq 1$, no characteristic value exists. Consider the expectation value of the magnitude

$$\langle m \rangle = \int_{m_\epsilon}^{m_L} m\, p(m)\mathrm{d}m \sim \int_{m_\epsilon}^{m_L} m^{-\alpha}\mathrm{d}m = \frac{m_L^{1-\alpha} - m_\epsilon^{1-\alpha}}{1-\alpha} \to \infty \begin{cases} \alpha < 1 \text{ and } m_L \to \infty \ , \\ \alpha > 1 \text{ and } m_\epsilon \to 0 \ . \end{cases} \quad (6)$$

Here m_L is the maximum event magnitude, which diverges with system size L, and m_ϵ is the smallest, "atomic scale", event size. If $\alpha \leq 1$, then $\langle m \rangle$ fails to exist and diverges with system size (note that $\langle m \rangle \to \ln m_L$ as $\alpha \to 1^-$). Similarly, if $\tau \leq 1$ then $\langle t \rangle \to \infty$ as system size increases and the system is also scale-free in the time domain. In this case the system is *scale-free* and has no characteristic event magnitude or duration; and any model of the system must also be scale-free and cannot depend on atomistic details. We consider such models not to be SOC.

We therefore consider systems to be self-organized critical provided that eqs. (4) and (5) are satisfied, and the first and higher moments of both the magnitude- and the duration probability densities diverge with system size.

From the modeling point of view, we expect that systems with finite first moments, that is, $\alpha > 1$, and/or $\tau > 1$, require "atomistic" details since the distributions then are dominated by the small scale and/or short time behavior, i.e., dominated by m_ϵ in (6) even if $m_L \to \infty$.

TABLE I: Scaling exponents for magnitude and duration

Distribution		Carpet	Rain [8]	Granular [9]	Earthquakes [11, 12]
$P(M > m) \sim m^{-\alpha}$	α	0.67	0.4	0.94	2/3
$P(T > t) \sim t^{-\tau}$	τ	0.96	0.6	1.08	2
$t \sim m^{\gamma}$	γ	0.69	0.7	0.87	1/3
$\alpha = \gamma\tau$	$\gamma\tau$	0.66	0.4	0.94	2/3

DISCUSSION

Table I collects values for the scaling exponents and tests the validity of the dynamic scaling relation eq. (5). In our experiments we have measured the exponents α, τ, and γ. We find that both $\alpha < 1$ and $\tau < 1$, and that the scaling relation $\alpha = \gamma\tau$ holds within experimental accuracy. We conclude that the stick-slip process studied is SOC.

The dynamic scaling relation may also be tested on other "avalanche phenomena" where size and duration are known to be power-law distributed such as the size and duration of rain showers [8], and the slips in a granular medium under shear [9]. We note that the granular slip experiments [9] have $\tau = 1.08$ and the system is therefore not scale free unless $\tau \leq 1$, which is within experimental error.

The exponents take different values for different systems. In the language of critical phenomena [15], we would say that stick-slip motion and rain showers belong to different universality classes, since the exponents differ. We expect that for each SOC universality class there exist a scale free dynamic model characterizing that class.

Earthquakes have long been recognized as resulting from stick-slip friction [17]. Earthquakes may represent an example of SOC [7, 18, 19]. The magnitude of earthquakes is characterized by their *seismic moments* M_0 [11], which has the Gutenberg-Richter distribution $P(M > M_0) \sim M_0^{-B}$ with the exponent $B = \alpha = 2/3$ [11, 12]. Implicitly it is assumed that M_0 is the only stochastic variable for earthquakes, other quantities such as duration t, and the area S are "mechanically" related to M_0. The duration is $t = L/V$, where V is the rupture velocity, and $L = S^{1/2}$; also $M_0 \propto S^{3/2}$. These scaling relations are consistent with observations [12]. It follows that $t \propto M_0^{1/3}$, i.e., $\gamma = 1/3$; and $\tau = 2$. Consequently, in our view, earthquakes are not SOC since they are not described by a joint probability density for magnitude and duration that is a generalized homogeneous function, and because $\tau > 1$. Models of earthquakes must include "microscopic" details for the duration, and be scale free for magnitudes, in contradiction with the mechanistic relation $t = M_0^{1/3}$.

The original paper [7] on SOC introduced a cellular automaton model for avalanches (with $\alpha = 0$ for a $(50)^2$ system). However, the exponents are not well determined since they depend strongly on system size [20]. For a $(2048)^2$ system $\alpha = 0.13$, $\gamma = 0.61$, and $\tau = 0.20$ consistent with eq. (5). The conservative OFC model [21] has exponents $\alpha = 0.25$, $\gamma = 0.42$, and $\tau = 0.52$ satisfying the scaling relation. Here duration was taken to be the number of cellular automaton updates required to end an avalanche. Both models are SOC with our definition. The models just discussed cannot be in the universality class of our stick-slip motion, or earthquakes, since they have too small values for α.

CONCLUSION

We conclude that stick-slip motion in the system we have studied exhibits power-law scaling that is well described by a joint probability density for magnitude and duration that has the form of a generalized homogeneous function. We derived a scaling relation between the dynamic exponents τ and γ, and the magnitude exponent α that takes the simple form $\alpha = \gamma\tau$. We have found other examples in natural and experimental systems that satisfy this scaling relation. We show that our analysis leads to open questions for earthquakes.

We have proposed that self-organized critical (SOC) systems are characterized by

- a joint probability density for magnitude and duration that is a generalized homogeneous function,
- with exponents $\alpha \leq 1$ and $\tau \leq 1$, and
- that satisfy the scaling relation $\alpha = \gamma\tau$.

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